

Stochastic extensions of symbols in Wiener spaces and heat operator.

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Abstract

The construction, in [AJN], of a pseudodifferential calculus analogous to the Weyl calculus, in an infinite dimensional setting, required the introduction of convenient classes of symbols.

In this article, we proceed with the study of these classes in order to establish, later on, the properties that a pseudodifferential calculus is expected to satisfy. The introduction and the study of a new class are rendered necessary in view of applications in QED.

We prove here that the symbols of both classes and the terms of their Taylor expansions admit stochastic extensions. We define, in this infinite dimensional setting, a semigroup H_t analogous to the heat semigroup, acting on the symbols belonging to both classes of symbols. The heat operator commutes with a second order operator similar to the Laplacian, which is its infinitesimal generator. For the class defined there, we give an expansion in powers of t of $H_t f$, according to the classes of symbols.

Keywords : stochastic extensions, heat operator, Wiener spaces, pseudodifferential calculus, symbol classes

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1 Introduction

This article follows [AJN] where a pseudodifferential Weyl calculus in an infinite dimensional setting has been developped, replacing \mathbb{R}^n by a probability space, the abstract Wiener space, denoted by B (one may consult [K] about this topic). This space is the completion of a real, separable, infinite dimensional Hilbert space H with respect to a convenient norm (called “measurable”), which is different from the canonical norm of H . We then have two complete spaces, one endowed with a scalar product and a symplectic form (on H^2),

the other one, with a probability measure $\mu_{B,t}$ generalizing the finite dimensional Gaussian measure, the positive parameter t representing the variance. This distribution of properties compels us to shift constantly from one space to the other, which is naturally not the case in the finite dimensional setting. Remark that, since the completion of H depends on the choice of the norm, it is not unique and we shall take advantage of it.

In [AJN], the Weyl calculus has been constructed for symbols belonging to a given class of symbols, $S_m(\mathcal{B}, \varepsilon)$, recalled in Definition 2.3. The symbols are functions defined on the Hilbert space H and satisfying partial differentiability conditions with respect to a fixed orthonormal basis \mathcal{B} , as well as estimates. These properties allow us to extend the symbols, in a certain sense, as functions defined on the Wiener space B . This is the notion of stochastic extension, recalled in Definition 2.2 and which is generally different from a continuity extension. The symbols and the calculus depend strongly on the chosen basis \mathcal{B} . Nevertheless, the basis is arbitrary and we shall see, moreover, that the analogue of the Laplace operator does not depend on \mathcal{B} , under precise conditions.

Some points, which are important in pseudodifferential analysis, have still to be solved. The construction has been completed in [AJN], but the covariance has been proved only in the most simple case. Beals characterization has been treated in [ALN], whereas the composition results have been obtained in a high (but finite) dimensional setting [AN1].

In the present article, we introduce another class of symbols, $S(Q_A)$, defined thanks to a quadratic form (Definition 3.6). Indeed, the first classes could be used in quantum electrodynamics but only under a truncature assumption (see [ALN2]), which the classes $S(Q_A)$ enable us to lift. The new classes $S(Q_A)$ do not depend on a basis either. We go further with the study of the classes $S_m(\mathcal{B}, \varepsilon)$, begun in [AJN], where only the properties absolutely necessary in view of the construction itself had been developed. In the same way, we prove comparable properties for the new classes. The aim is to define a semigroup of operators similar to the heat operator and to state the properties which will be needed to treat the composition of operators.

We first generalize a result of [AJN] about the existence of stochastic extensions for the classes $S_m(\mathcal{B}, \varepsilon)$ (Proposition 3.1) and prove that their symbols are Frechet-differentiable for sufficiently large m . We then prove the existence of stochastic extensions for symbols in the new classes $S(Q_A)$ (Proposition 3.10). Next we study the Taylor's expansions for the symbols in both classes, in order to give expansions for the regular terms and for the rest. One states there stochastic extension properties for unbounded functions (Proposition 4.11), for which, in certain cases, the Weyl calculus has been defined by other means in [AJN]. This will prove indispensable to treat the polynomial terms in Taylor's expansions.

The next step is the construction of a semigroup similar to the heat semigroup, denoted by $(H_t)_{t \geq 0}$. For bounded Borel functions defined on the Wiener space B itself, it is a classical notion, to which [G-4] is almost entirely devoted and which is still being studied ([HA]). It is given by

$$\forall x \in B, \quad H_t f(x) = \int_B f(x+y) d\mu_{B,t}(y)$$

with our notations. When f is bounded and uniformly continuous, $H_t f$ converges uniformly to f when t converges to 0. If, moreover, f is Lipschitz continuous on B , $H_t f$ has further differentiability properties. But our construction requires this notion for symbols f defined on the initial Hilbert space H , which is slightly complicated since H is $\mu_{B,t}$ -negligible in B . We then set:

$$\forall x \in H, \quad H_t f(x) = \int_B \tilde{f}(x+y) d\mu_{B,t}(y),$$

where \tilde{f} is a stochastic extension of f in a certain sense. For a completion of H given by an arbitrary measurable norm, this function \tilde{f} is not necessarily continuous or uniformly continuous. But we prove, for both classes, the existence of a precise completion B_A of H in which the stochastic extensions have useful topological properties (Propositions 3.15 and 4.7). This allows us to use the classical theory of [G-4] and [K]. The extension B_A may be different from the extension B initially chosen and is used temporary. Of course, one checks that the integral defining $H_t f(x)$ does not depend on the chosen Wiener space completing H .

The main results of this article are Theorems 5.9, 5.17, which establish, for the classes $S_m(\mathcal{B}, \varepsilon)$ as for the classes $S(Q_A)$, the existence of a Laplacian commuting with the heat operator and which is its infinitesimal operator. This result will play an important part in the composition of symbols. Let us stress the following

expansion for $f \in S(Q_A)$,

$$H_t f = f + \sum_{k=1}^N \frac{t^k}{k!} \left(\frac{1}{2} \Delta \right)^k f + t^{N+1} R_N(t),$$

where the rest satisfies estimates independent of t . To conclude, we give, for the class $S(Q_A)$, an invariance property (Proposition 5.16) which will be useful to prove a covariance result.

Section 2 recalls the indispensable notions about Wiener space and Wiener measure. Then it gives the vital definitions and results about the Weyl calculus in an infinite dimensional setting. Section 3 recalls and states more precisely the results about stochastic extensions for the classes $S_m(\mathcal{B}, \varepsilon)$ of [AJN]. It proves similar results in the case of products of scalar products and for the classes $S(Q_A)$, which are defined at this point. It brings up the alternative definition of the Weyl calculus, as a quadratic form, which enables us to use unbounded symbols. In Section 4 we prove the Frechet-differentiability of the symbols in the classes $S_m(\mathcal{B}, \varepsilon)$ and we extend stochastically the Taylor's expansions of symbols of both classes. Section 5 defines the heat operator H_t for functions initially defined on the Hilbert space (and which it is impossible to integrate on the Wiener space without an extension). We establish the semigroup property for both classes, together with useful properties of H_t (infinitesimal generator, commutation). The technical results about classical integration are stated in the appendix (section 6).

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2 The Weyl calculus on a Wiener space

The construction of the Wiener space may be found in [G-1, G-2, G-3, K]. The Weyl calculus on a Wiener space has been developed in [AJN]. We just recall here the notions which are necessary to read the present article.

The abstract Wiener space (i, H, B) is a triple where H is a real, separable, infinite dimensional Hilbert space, B is a Banach space containing H and i is the canonical injection (which is not always mentioned). Moreover, H is continuously embedded in B as a dense subspace. Sometimes, B itself is called the Wiener space, as opposed to H , when no confusion is possible. One denotes by $\langle \cdot \rangle$ (or sometimes \cdot) and $\| \cdot \|$ the scalar product and the norm on H and by $\| \cdot \|$ the norm on B .

One identifies H with its dual space, so that $B' \subset H \subset B$, each space being a dense subspace of the following one. One denotes by $\mathcal{F}(X)$ the set of all finite dimensional subspaces of a vector space X . If $E \in \mathcal{F}(H)$, one denotes by π_E the orthogonal projection of H onto E .

It is impossible to extend to H itself the Gaussian measure which is naturally defined on its finite dimensional subspaces. Nevertheless, if the norm $\| \cdot \|$ of B has a property called measurability (see Definition 4.4 Chap 1 [K] or [G-1]), one can construct a Gaussian measure on the Borel σ -algebra of B . Let us denote by

$$d\mu_{\mathbb{R}^n, h}(x) = (2\pi h)^{-n/2} e^{-\frac{1}{2h} \sum_{i=1}^n x_i^2} d\lambda(x_1, \dots, x_n)$$

the Gaussian measure with variance $h > 0$ on \mathbb{R}^n . A cylinder of B is a set of the form

$$\mathcal{C} = \{x \in B : (y_1(x), \dots, y_n(x)) \in A\}, \quad (1)$$

where n is a positive integer, y_1, \dots, y_n are elements of B' and A is a Borel set of \mathbb{R}^n . One defines the measure of this cylinder setting

$$\mu_{B, h}(\mathcal{C}) = \int_A d\mu_{\mathbb{R}^n, h}(x) \quad (2)$$

in case the family (y_1, \dots, y_n) is orthonormal with respect to the scalar product of H , which can always be assumed. The parameter h represents the variance of the Gaussian measure and can also be considered as a semiclassical parameter in the Weyl calculus. One can prove that this measures extends as a probability measure, still denoted by $\mu_{B, h}$, on the σ -algebra generated by the cylinders of B , which is the Borel σ -algebra of B . The same definition, but starting from cylinders of H , yields a pseudomeasure which is not σ -additive.

If $E \in \mathcal{F}(B')$ has dimension n , one can identify E and \mathbb{R}^n by choosing a basis, orthonormal with respect to the scalar product of H and thus define a measure $\mu_{E, h}$ on E . For every function $\varphi \in L^1(E, \mu_{E, h})$, the transfer theorem

$$\int_B \varphi \circ P_E(x) d\mu_{B, h}(x) = \int_E \varphi(u) d\mu_{E, h}(u). \quad (3)$$

If y is an element of B' , it can be considered as a random variable on B . If y is not zero one sees, using (2), that, for every Borel set A of \mathbb{R} ,

$$\mu_{B,h}(y \in A) = \int_A e^{-\frac{y^2}{2h|y|^2}} (2\pi h|y|^2)^{-1/2} dv,$$

which means that y has the normal distribution $\mathcal{N}(0, \sigma^2 = h|y|^2)$ [K]. Up to the factor \sqrt{h} , there exists an isometry from $(B', | \cdot |)$ in $L^2(B, \mu_{B,h})$. It can be extended as an isometry from H in $L^2(B, \mu_{B,h})$ and one denotes by ℓ_a the image of an element a of H . If $a \in B'$, $\ell_a = a$ is a linear application but if $a \in H$, ℓ_a is only defined $\mu_{B,h}$ -almost everywhere and is not necessarily linear. However, $\ell_a(-x) = -\ell_a(x)$ and $\ell_a(x+y) = \ell_a(x) + a \cdot y$ for $y \in H$.

If $E \in \mathcal{F}(H)$ has an orthonormal basis (e_1, \dots, e_n) , one sets, for $x \in B$,

$$\tilde{\pi}_E(x) = \sum_{j=1}^n \ell_{u_j}(x) u_j, \quad (4)$$

in keeping with the projection. Then, for all $a \in H$,

$$a \cdot \tilde{\pi}_E(x) = \ell_{\pi_E(a)}(x). \quad (5)$$

The functions ℓ_a satisfy the following identities, recalled in [AJN]. If $a = u + iv$, with u and v in H , then

$$\int_B e^{\ell_a(x)} d\mu_{B,h}(x) = e^{h\frac{a^2}{2}}. \quad (6)$$

One has set $a^2 = |u|^2 - |v|^2 + 2iu \cdot v$. For all a in H and for all $p \geq 1$:

$$\int_B |\ell_a(x)|^p d\mu_{B,h}(x) = \frac{(2h)^{p/2}}{\sqrt{\pi}} |a|^p \Gamma\left(\frac{p+1}{2}\right). \quad (7)$$

Setting

$$K(p) = 2^{1/2} \pi^{-1/2p} \left(\Gamma\left(\frac{p+1}{2}\right) \right)^{1/p}, \quad (8)$$

one can write that $\|\ell_a\|_{L^p(B, \mu_{B,h})} = K(p) h^{1/2} |a|$. Notice that $K(2) = 1$. One sees, too, that for all a and b in H ,

$$\int_B e^{\ell_b(u)} |\ell_a(u)|^p d\mu_{B,h}(u) = e^{h\frac{|b|^2}{2}} \int_{\mathbb{R}} |\sqrt{h}|a|v + ha \cdot b|^p d\mu_{\mathbb{R},1}(v). \quad (9)$$

Let us recall the theorem of Wick :

Theorem 2.1. Wick *Let u_1, \dots, u_{2p} be vectors of H ($p \geq 1$). Let $h > 0$. Then one has*

$$\int_B \ell_{u_1}(x) \dots \ell_{u_{2p}}(x) d\mu_{B,h}(x) = h^p \sum_{(\varphi, \psi) \in S_p} \prod_{j=1}^p \langle u_{\varphi(j)}, u_{\psi(j)} \rangle \quad (10)$$

where S_p is the set of all couples (φ, ψ) of injections from $\{1, \dots, p\}$ into $\{1, \dots, 2p\}$ such that:

1. For all $j \leq p$, $\varphi(j) < \psi(j)$.
2. The sequence $(\varphi(j))_{(1 \leq j \leq p)}$ is an increasing sequence.

The measure $\mu_{B,h}$ transforms, under translation of a vector a belonging to H , into another measure which is absolutely continuous with respect to the former one. More precisely, for all $g \in L^1(B, \mu_{B,h})$, one has, for all a in H :

$$\int_B g(x) d\mu_{B,h}(x) = e^{-\frac{1}{2h}|a|^2} \int_B g(x+a) e^{-\frac{1}{h}\ell_a(x)} d\mu_{B,h}(x). \quad (11)$$

But if the translation vector a belongs to B , or if the variance parameter h changes, both measures are mutually singular.

The Weyl calculus on the Wiener space has been constructed in two different ways. One of the constructions is rather similar to the classical definition, in that it relies on classes of symbols which satisfy differentiability conditions and it yields operators which are bounded on a L^2 space. We will work in this frame most of the time. We do not have, though, an integral definition of $Op(f)u$, neither on H nor on B . The symbols are functions defined on H^2 by Definition 2.3. It is possible - and necessary - to extend them to functions defined on B^2 according to the definition below. This notion is classical in the theory of Wiener spaces (see [G-1, G-2, G-3], [RA], [K]).

Definition 2.2. Let (i, H, B) be an abstract Wiener space such that $B' \subset H \subset B$ (The inclusion i will be omitted). Let h be a positive real number.

1. A Borel function f , defined on H , admits a stochastic extension \tilde{f} with respect to the measure $\mu_{B,h}$ if, for every increasing sequence (E_n) in $\mathcal{F}(H)$, whose union is dense in H , the sequence of functions $f \circ \tilde{\pi}_{E_n}$ (where $\tilde{\pi}_{E_n}$ is defined by (4)) converges in probability with respect to the measure $\mu_{B,h}$ to \tilde{f} . In other words, if, for every $\delta > 0$,

$$\lim_{n \rightarrow +\infty} \mu_{B,h} \left(\left\{ x \in B, \quad |f \circ \tilde{\pi}_{E_n}(x) - \tilde{f}(x)| > \delta \right\} \right) = 0. \quad (12)$$

2. A function f admits a stochastic extension \tilde{f} in $L^p(B, \mu_{B,h})$ ($1 \leq p < \infty$) if, for every increasing sequence (E_n) in $\mathcal{F}(H)$, whose union is dense in H , the functions $f \circ \tilde{\pi}_{E_n}$ are in $L^p(B, \mu_{B,h})$ and if the sequence $f \circ \tilde{\pi}_{E_n}$ converges in $L^p(B, \mu_{B,h})$ to \tilde{f} .

One defines likewise the stochastic extension of a function on H^2 to a function on B^2 .

One can check, for example thanks to (7), that ℓ_a is the stochastic extension of the scalar product with a and that $\tilde{\pi}_E$ is the stochastic extension of π_E in L^p . The stochastic extension can be obtained in a more topological manner (see [K], chap. 1, par. 6). Let us draw attention to a result about extensions of holomorphic functions (Theorem 8.8 [AJN]), obtained by martingale methods and proving a property announced by [K-R].

The symbol classes used in [AJN] share derivability properties and estimates with the classes of the Calderón-Vaillancourt Theorem:

Definition 2.3. Let (i, H, B) be an abstract Wiener space such that $B' \subset H \subset B$. Let $\mathcal{B} = (e_j)_{(j \in \Gamma)}$ be a Hilbert basis of H , each vector belonging to B' , indexed by a countable set Γ . Set $u_j = (e_j, 0)$ and $v_j = (0, e_j)$ ($j \in \Gamma$). A multi-index is a map (α, β) from Γ into $\mathbb{N} \times \mathbb{N}$ such that $\alpha_j = \beta_j = 0$ except for a finite number of indices. Let M be a nonnegative real number, m a nonnegative integer and $\varepsilon = (\varepsilon_j)_{(j \in \Gamma)}$ a family of nonnegative real numbers. One denotes by $S_m(\mathcal{B}, M, \varepsilon)$ the set of bounded continuous functions $F : H^2 \rightarrow \mathbb{C}$ satisfying the following condition. For every multi-index (α, β) of depth m , that is to say such that $0 \leq \alpha_j \leq m$ and $0 \leq \beta_j \leq m$ for all $j \in \Gamma$, the following derivative

$$\partial_u^\alpha \partial_v^\beta F = \left[\prod_{j \in \Gamma} \partial_{u_j}^{\alpha_j} \partial_{v_j}^{\beta_j} \right] F \quad (13)$$

is well defined, continuous on H^2 and satisfies, for every (x, ξ) in H^2

$$\left| \left[\prod_{j \in \Gamma} \partial_{u_j}^{\alpha_j} \partial_{v_j}^{\beta_j} \right] F(x, \xi) \right| \leq M \prod_{j \in \Gamma} \varepsilon_j^{\alpha_j + \beta_j}. \quad (14)$$

One recalls the following very useful property, stated in the proof of Proposition 4.14 of [AJN]. If ε is square summable, every function F in $S_1(\mathcal{B}, M, \varepsilon)$ verifies, for all X and V in H^2 , a Lipschitz condition:

$$|F(X + V) - F(X)| \leq M|V|\sqrt{2} \left[\sum_{j \in \Gamma} \varepsilon_j^2 \right]^{1/2}. \quad (15)$$

It is more convenient to represent classes of symbols as vector spaces.

Definition 2.4. Let ε be a sequence of positive real numbers and let $m \in \mathbb{N}$. One sets $S_m(\mathcal{B}, \varepsilon) = \bigcup_{M \geq 0} S_m(\mathcal{B}, M, \varepsilon)$. For $F \in S_m(\mathcal{B}, \varepsilon)$ one sets $\|F\|_{m, \varepsilon} = \inf\{M \geq 0 : F \in S_m(\mathcal{B}, M, \varepsilon)\}$.

Remark that $S_m(\mathcal{B}, \varepsilon)$, equipped with $\|\cdot\|_{m, \varepsilon}$, is a Banach space. Setting $S^\infty(\mathcal{B}, \varepsilon) = \bigcap_{m=0}^\infty S_m(\mathcal{B}, \varepsilon)$, one can, classically, define a distance by $d(F, G) = \sum_{m=0}^\infty 2^{-m} \frac{\|F - G\|_{m, \varepsilon}}{1 + \|F - G\|_{m, \varepsilon}}$. Then $(S^\infty(\mathcal{B}, \varepsilon), d)$ is complete.

An alternative construction of the Weyl calculus uses an analogue of the Wigner function in order to associate a quadratic form with a function \tilde{F} defined, this time, on B^2 . This quadratic form is applied to cylindrical functions, depending on a finite number of variables. Let us only recall that this construction

requires of \tilde{F} to belong to $L^1(B^2, \mu_{B^2, h/2})$ and to be such that there exists a nonnegative integer m such that

$$N_m(\tilde{F}) = \sup_{Y \in H^2} \frac{\|\tilde{F}(\cdot + Y)\|_{L^1(B^2, \mu_{B^2, h/2})}}{(1 + |Y|)^m} < +\infty. \quad (16)$$

This norm is finite if the function \tilde{F} is bounded or if it is a polynomial expression of degree m with respect to functions $(x, \xi) \rightarrow \ell_a(x) + \ell_b(\xi)$, with a and b in H , as we shall see in Subsection 3.3.

These approaches complement one another. The most classical enables us to work on L^2 spaces on B , but the symbol has to be bounded, the other one allows us to use non bounded symbols, but the domain of the quadratic forms contains only cylindrical functions. Both definitions coincide under certain conditions (Theorem 1.4 [AJN]).

3 Stochastic extensions

3.1 Stochastic extensions of symbols in $S_m(\mathcal{B}, \varepsilon)$

We first generalize a proposition stated in [AJN] (Proposition 8.4) in the case when $p = 1$.

Proposition 3.1. *Let F be a function in $S_1(\mathcal{B}, \varepsilon)$, with respect to a Hilbert basis $\mathcal{B} = (e_j)_{(j \in \Gamma)}$, where the sequence $(\varepsilon_j)_{(j \in \Gamma)}$ is summable. Then, for every positive h and every $q \in [1, +\infty[$, F admits a stochastic extension in $L^q(B^2, \mu_{B^2, h})$.*

Moreover, for all $h_0 > 0$ and $q_0 \in]1, +\infty[$, there exists a function \tilde{F} which is the stochastic extension of F in $L^q(B^2, \mu_{B^2, h})$ for all $h \in]0, h_0]$ and $q \in [1, q_0]$.

For any $E \in \mathcal{F}(H^2)$, we then have the inequality : $\forall (h, q) \in]0, h_0] \times [1, q_0]$,

$$\|F \circ \tilde{\pi}_E - \tilde{F}\|_{L^q(B^2, \mu_{B^2, h})} \leq \|F\|_{1, \varepsilon} K(q) h^{1/2} \sum_{j=1}^{\infty} \varepsilon_j \left(|u_j - \pi_E(u_j)| + |v_j - \pi_E(v_j)| \right). \quad (17)$$

Proof. Let (E_n) be an increasing sequence of $\mathcal{F}(H^2)$, whose union is dense in H^2 . For all m and n such that $m < n$, let S_{mn} be the orthogonal supplement of E_m in E_n . We can state an inequality analogous to the inequality (120) of [AJN]:

$$\|F \circ \tilde{\pi}_{E_m} - F \circ \tilde{\pi}_{E_n}\|_{L^q(B^2, \mu_{B^2, h})} \leq \|F\|_{1, \varepsilon} K(q) h^{1/2} \sum_{j=1}^{\infty} \varepsilon_j \left(|\pi_{S_{mn}}(u_j)| + |\pi_{S_{mn}}(v_j)| \right). \quad (18)$$

Indeed, one just needs to replace L^1 by L^q in the original proof, since the only changes take place in the explicit L^q norms of the ℓ_a functions appearing there. This inequality proves that $F \circ \tilde{\pi}_{E_m}$ is a Cauchy sequence in $L^q(B^2, \mu_{B^2, h})$ and one can verify that the limit does not depend on the sequence (E_n) .

To construct a representant of the stochastic extension common to all $(h, q) \in]0, h_0] \times [1, q_0]$, let us assume that (E_n) is an increasing sequence of elements of $\mathcal{F}(B')$. The right term of (18) is smaller than an expression $C(m, n)$ which depends, neither on h , nor on q . This allows us to construct an increasing sequence $(n_i)_i$ satisfying $C(n_{i+1}, n_i) < 2^{-i-1}$ and a sequence of functions $(F_N)_N$ defined by

$$F_N := F \circ \tilde{\pi}_{E_{n_1}} + \sum_{j=1}^N \left(F \circ \tilde{\pi}_{E_{n_{j+1}}} - F \circ \tilde{\pi}_{E_{n_j}} \right) \text{ on } B^2,$$

exactly as in the classical proof of the Riesz-Fisher Theorem. The functions F_N are defined everywhere on B^2 and independent of (h, q) . The limit \tilde{F} of this sequence is the representant we are looking for and it takes finite values on a subset of B^2 whose $\mu_{B^2, h}$ -measure is 1 for all $h \leq h_0$.

The last inequality is a consequence of (18), with $E_m = E$ and letting n grow to infinity. \square

Corollary 3.2. *Let h_0 be a positive real number. There exists a function \tilde{F} which is the stochastic extension of F in $L^q(B^2, \mu_{B^2, h})$ for all $h \in]0, h_0]$ and all $q \geq 1$. Inequality (17) still holds.*

Proof. Denote by \tilde{F}_2 (resp. \tilde{F}_n) the stochastic expansion given by Proposition 3.1, for $h \in]0, h_0]$ and $q \in [1, 2]$ (resp. $q \in [1, n]$). Let (E_s) be an increasing sequence of $\mathcal{F}(H^2)$, whose union is dense in H^2 . One then has, for $q \leq 2$,

$$\lim_{s \rightarrow \infty} \|F \circ \tilde{\pi}_{E_s} - \tilde{F}_2\|_{q, h} = 0, \quad \lim_{s \rightarrow \infty} \|F \circ \tilde{\pi}_{E_s} - \tilde{F}_n\|_{q, h} = 0.$$

Consequently $\tilde{F}_2 = \tilde{F}_n$ $\mu_{B^2, h}$ almost everywhere. It follows that $\tilde{F}_2 \in L^q(B^2, \mu_{B^2, h})$ for $q \leq n$ and that the convergence is true. To obtain the inequality one similarly replaces \tilde{F}_n by \tilde{F}_2 . \square

Let us state another consequence of the proof of Proposition 8.4 of [AJN]:

Corollary 3.3. *If $F \in S_1(\mathcal{B}, \varepsilon)$ where ε is summable, then for all $h > 0$ and all $p \in [1, +\infty[$,*

$$|\tilde{F}| \leq \|F\|_{1, \varepsilon} \quad \mu_{B^2, h} - \text{a.s.}, \quad \|\tilde{F}\|_{L^p(B^2, \mu_{B^2, h})} \leq \|F\|_{1, \varepsilon}.$$

Let us denote by \mathcal{P} the operator associating, with a function in $S_1(\mathcal{B}, \varepsilon)$, its stochastic extension in $L^p(B^2, \mu_{B^2, t})$. This operator is thus linear, bounded and its norm is smaller than 1.

Proof. For every increasing sequence $(E_n)_n$ of $\mathcal{F}(H^2)$, whose union is dense in H^2 , the sequence $(F \circ \tilde{\pi}_{E_n})_n$ converges to \tilde{F} in L^p . Since $|F \circ \tilde{\pi}_{E_n}|$ is smaller than $\|F\|_{1, \varepsilon} \mu_{B^2, h}$ almost everywhere on B^2 (on the domain where $\tilde{\pi}_{E_n}$ is defined or on B^2 if $E_n \subset B'$), so is $|\tilde{F}|$. Moreover, $\|F \circ \tilde{\pi}_{E_n}\|_{L^p(B^2, \mu_{B^2, h})} \leq \|F\|_{1, \varepsilon}$ and letting n grow to infinity yields $\|\tilde{F}\|_{L^p(B^2, \mu_{B^2, h})} \leq \|F\|_{1, \varepsilon}$. \square

Corollary 3.4. *If $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 1$ and ε summable, if \tilde{F} is the stochastic extension in the L^p given above, if $Y \in H^2$, then $\tau_Y F$ admits $\tau_Y \tilde{F}$ as a stochastic extension in the $L^p(B^2, \mu_{B^2, h})$ for $h > 0$ and $p \in [1, +\infty[$.*

Proof. Let (E_j) be an increasing sequence of $\mathcal{F}(H^2)$, whose union is dense in H^2 . If we denote by an index p the $L^p(B^2, \mu_{B^2, h})$ norm, we obtain that

$$\|\tau_Y \tilde{F} - (\tau_Y F) \circ \tilde{\pi}_{E_j}\|_p \leq \|\tau_Y \tilde{F} - \tau_Y(F \circ \tilde{\pi}_{E_j})\|_p + \|\tau_Y(F \circ \tilde{\pi}_{E_j}) - (\tau_Y F) \circ \tilde{\pi}_{E_j}\|_p.$$

For all $p' > p$, the inequality

$$\|\tau_Y \tilde{F} - \tau_Y(F \circ \tilde{\pi}_{E_j})\|_p = \left(\int_{B^2} |\tilde{F} - F \circ \tilde{\pi}_{E_j}|^p(X) e^{\frac{1}{h} \ell_Y(X)} d\mu_{B^2, h}(X) e^{-\frac{1}{2h} |Y|^2} \right)^{1/p} \leq \|\tilde{F} - F \circ \tilde{\pi}_{E_j}\|_{p'} e^{\frac{|Y|^2}{2h(p'-p)}}$$

holds true, thanks to the translation change of variables (11), to Hölder's inequality and to the formula (6). The second term converges to 0 as well, thanks to (15), which enables us to give an upper bound of

$$|F(\tilde{\pi}_{E_j}(X + Y)) - F(\tilde{\pi}_{E_j}(X) + Y)| = |F(\tilde{\pi}_{E_j}(X) + \pi_{E_j}(Y)) - F(\tilde{\pi}_{E_j}(X) + Y)|.$$

\square

Remark 3.5. *The result above holds for every globally Lipschitz continuous function F admitting a stochastic extension \tilde{F} in L^p for every $p \in [1, +\infty[$ and $h > 0$.*

3.2 Symbol classes defined thanks to a quadratic form

Definition 3.6. *Let A be a linear, selfadjoint, nonnegative, trace class application on a Hilbert space H . For all $x \in H$ one sets $Q_A(x) = \langle Ax, x \rangle$. Let $S(Q_A)$ be the class of all functions $f \in C^\infty(H)$ such that there exists $C(f) > 0$ satisfying:*

$$\begin{aligned} \forall x \in H, \quad |f(x)| &\leq C(f), \\ \forall m \in \mathbb{N}^*, \quad \forall x \in H, \quad \forall (U_1, \dots, U_m) \in H^m, \quad |(d^m f)(x)(U_1, \dots, U_m)| &\leq C(f) \prod_{j=1}^m Q_A(U_j)^{\frac{1}{2}}. \end{aligned} \quad (19)$$

The smallest constant $C(f)$ such that (19) holds is denoted by $\|f\|_{Q_A}$.

Notice that $S(Q_A)$, equipped with the norm $\|\cdot\|_{Q_A}$, is a Banach space. One can also check that, if A and B satisfy the conditions of Definition 3.6, their product belongs to $S(Q_{2(A+B)})$ with

$$\|fg\|_{Q_{2(A+B)}} \leq \|f\|_{Q_A} \|g\|_{Q_B}.$$

Moreover, if A is as above but defined on H^2 , the class $S(Q_A)$ is included in a class $S_\infty(\mathcal{B}, \varepsilon)$ for any orthonormal basis $\mathcal{B} = (e_j)$ of H , with $\varepsilon_j = \max(Q_A(e_j, 0)^{1/2}, Q_A(0, e_j)^{1/2})$. Since the sequence ε is only square summable, the existence results for the stochastic extensions must be obtained otherwise.

Remark 3.7. The constant $C(f)$ in the Definition 3.6 could depend on the order m . Some results are still valid with a less restrictive class satisfying

$$\forall m \in \mathbb{N}^*, \exists C_m = C_m(f) : \forall x \in H, \forall (U_1, \dots, U_m) \in H^m, |(d^m f)(x)(U_1, \dots, U_m)| \leq C_m(f) \prod_{j=1}^m Q_A(U_j)^{\frac{1}{2}}.$$

Lemma 3.8. For $E \in \mathcal{F}(H)$ and $h > 0$, $y \mapsto Q_A(\tilde{\pi}_E(y))^{\frac{1}{2}}$ belongs to $L^p(B, \mu_{B,h})$ for all $p \in [1, +\infty[$. More precisely, if (u_j) is a Hilbert basis of H whose vectors are eigenvectors of A (or belong to $\text{Ker}(A)$) and if one denotes by λ_j the corresponding eigenvalues, one obtains

$$\|Q_A^{\frac{1}{2}} \circ \tilde{\pi}_E\|_{L^p(B, \mu_{B,h})} \leq C(p) \left(\sum_0^\infty \lambda_j |\pi_E(u_j)|^{\alpha(p)} \right)^{1/\alpha(p)} h^{\frac{1}{2}},$$

with

$$\begin{aligned} C(p) &= K(p) \left(\sum_0^\infty \lambda_j \right)^{\frac{1}{2} - \frac{1}{p}} & \alpha(p) &= p \quad \text{for } p > 2 \\ C(p) &= 1 & \alpha(p) &= 2 \quad \text{for } p \leq 2, \end{aligned} \tag{20}$$

the constant $K(p)$ being defined by (8).

Proof. By decomposing A on its eigenvector basis, one obtains that

$$Q_A(\tilde{\pi}_E(y)) = \sum_{j=0}^\infty \lambda_j (u_j \cdot \tilde{\pi}_E(y))^2 = \sum_{j=0}^\infty \lambda_j (\ell_{\pi_E(u_j)})^2,$$

using (5). For $p = 2$ it suffices to integrate this equality and to use (7). For $p > 2$, one uses Jensen's inequality for a probability measure on \mathbb{N} . Set $S = \sum_0^\infty \lambda_j$. One then has

$$Q_A(\tilde{\pi}_E(y))^{\frac{p}{2}} = \left(\sum_{j=0}^\infty \frac{\lambda_j}{S} S (\ell_{\pi_E(u_j)})^2 \right)^{\frac{p}{2}} \leq \sum_{j=0}^\infty \frac{\lambda_j}{S} (S (\ell_{\pi_E(u_j)})^2)^{\frac{p}{2}},$$

and it remains to integrate. Finally, for $p \in [1, 2[$, one applies Hölder's inequality. \square

Remark 3.9. One can give an upper bound for $\|Q_A \circ \tilde{\pi}_E\|_{L^p(B, \mu_{B,h})}$, which does not depend on E :

$$\|Q_A^{\frac{1}{2}} \circ \tilde{\pi}_E\|_{L^p(B, \mu_{B,h})} \leq C(p) \left(\sum_0^\infty \lambda_j \right)^{1/\alpha(p)} h^{\frac{1}{2}}.$$

One can prove the following result.

Proposition 3.10. Let $h > 0$ and let $p \in [1, +\infty[$. Every function f belonging to $S(Q_A)$ admits a stochastic extension \tilde{f} in $L^p(B, \mu_{B,h})$. The function \tilde{f} is bounded $\mu_{B,h}$ almost everywhere by $\|f\|_{Q_A}$. Moreover, for all $E \in \mathcal{F}(H)$,

$$\|f \circ \tilde{\pi}_E - \tilde{f}(x)\|_{L^p(B, \mu_{B,h})} \leq C(p) h^{\frac{1}{2}} \|f\|_{Q_A} \left(\sum_{j \geq 0} \lambda_j |\pi_E(u_j) - u_j|^{\alpha(p)} \right)^{1/\alpha(p)}, \tag{21}$$

with the notations of Lemma 3.8.

Proof. Let (E_n) be an increasing sequence of $\mathcal{F}(H)$, whose union is dense in H . Let f be in $S(Q_A)$. Let m and n be such that $m < n$. Let S_{mn} be an orthogonal supplement of E_m in E_n . Then

$$f(\tilde{\pi}_{E_n}(x)) - f(\tilde{\pi}_{E_m}(x)) = \int_0^1 (df)(\tilde{\pi}_{E_m}(x) + \theta \tilde{\pi}_{S_{mn}}(x))(\tilde{\pi}_{S_{mn}}(x)) d\theta.$$

Hence

$$|f(\tilde{\pi}_{E_n}(x)) - f(\tilde{\pi}_{E_m}(x))| \leq \|f\|_{Q_A} \int_0^1 Q_A(\tilde{\pi}_{S_{mn}}(x))^{\frac{1}{2}} d\theta = \|f\|_{Q_A} Q_A(\tilde{\pi}_{S_{mn}}(x))^{\frac{1}{2}}.$$

This implies that

$$\|f \circ \tilde{\pi}_{E_n} - f \circ \tilde{\pi}_{E_m}\|_{L^p(B, \mu_{B,h})} \leq \|f\|_{Q_A} \|Q_A^{\frac{1}{2}} \circ \tilde{\pi}_{S_{mn}}\|_{L^p(B, \mu_{B,h})}.$$

Using the preceding Lemma 3.8, one gets that

$$\|f \circ \tilde{\pi}_{E_n} - f \circ \tilde{\pi}_{E_m}\|_{L^p(B, \mu_{B,h})} \leq C(p) h^{\frac{1}{2}} \|f\|_{Q_A} \left(\sum_{j \geq 0} \lambda_j |\pi_{S_{mn}}(u_j)|^{\alpha(p)} \right)^{1/\alpha(p)}.$$

The right term converges to 0 when m grows to infinity, according to the dominated convergence Theorem. Indeed, for all j , $|\pi_{S_{mn}}(u_j)|$ converges to 0 when m grows to infinity, $|\pi_{S_{mn}}(u_j)|^{\alpha(p)} \leq 1$ and the series $\sum \lambda_j$ converges. The sequence $(f(\tilde{\pi}_{E_n}))_n$ is therefore a Cauchy sequence in $L^p(B, \mu_{B,h})$. One can verify that its limit, in $L^p(B, \mu_{B,h})$, does not depend on the sequence (E_n) . Since the function $|f \circ \tilde{\pi}_{E_n}|$ is almost everywhere smaller than $\|f\|_{Q_A}$, so is its limit. Finally, taking $E = E_m$ in one of the above inequalities and letting n converge to infinity yields (21). \square

Remark 3.11. This result holds true for the class of Remark 3.7, with $\max(C_0(f), C_1(f))$ instead of $\|f\|_{Q_A}$ in the estimates.

Proposition 3.12. Let $h > 0$, $p \in [1, +\infty[$. Let k be a positive integer and let x be a fixed point in H . Set $S = \sum \lambda_j$. The function $y \mapsto d^k f(x) \cdot y^k$ defined on H admits a stochastic extension in $L^p(B, \mu_{B,h})$. Moreover, for all $E \in \mathcal{F}(H)$,

$$\|d^k f(x) \cdot \tilde{\pi}_E(y)^k - \mathcal{P}(y \mapsto d^k f(x) \cdot y^k)\|_p \leq k \|f\|_{Q_A} C(pk)^k S^{\frac{k-1}{\alpha(pk)}} h^{\frac{k}{2}} \left(\sum \lambda_s |\pi_E(u_s) - u_s|^{\alpha(pk)} \right)^{\frac{1}{\alpha(pk)}}.$$

Proof. Let $E, F \in \mathcal{F}(H)$ with $E \subset F$. For all $y \in B$, one has

$$\begin{aligned} d^k f(x) \cdot \tilde{\pi}_E(y)^k - d^k f(x) \cdot \tilde{\pi}_F(y)^k &= \sum_{j=1}^k d^k f(x) (\tilde{\pi}_E(y)^j, \tilde{\pi}_F(y)^{k-j}) - d^k f(x) (\tilde{\pi}_E(y)^{j-1}, \tilde{\pi}_F(y)^{k-j+1}) \\ &= \sum_{j=1}^k d^k f(x) (\tilde{\pi}_E(y)^{j-1}, \tilde{\pi}_E(y) - \tilde{\pi}_F(y), \tilde{\pi}_F(y)^{k-j}). \end{aligned}$$

Using Definition 3.6, one deduces that

$$|d^k f(x) \cdot \tilde{\pi}_E(y)^k - d^k f(x) \cdot \tilde{\pi}_F(y)^k| \leq \sum_{j=1}^k \|f\|_{Q_A} Q_A^{\frac{j-1}{2}}(\tilde{\pi}_E(y)) Q_A^{\frac{k-j}{2}}(\tilde{\pi}_F(y)) Q_A^{\frac{1}{2}}(\tilde{\pi}_E(y) - \tilde{\pi}_F(y)).$$

Using Hölder's inequality and Remark 3.9, one obtains that

$$\begin{aligned} &\|d^k f(x) \cdot \tilde{\pi}_E(y)^k - d^k f(x) \cdot \tilde{\pi}_F(y)^k\|_p \\ &\leq \sum_{j=1}^k \|f\|_{Q_A} \|Q_A^{\frac{1}{2}} \circ \tilde{\pi}_E\|_{pk}^{j-1} \|Q_A^{\frac{1}{2}} \circ \tilde{\pi}_F\|_{pk}^{k-j} \|Q_A^{\frac{1}{2}} \circ (\tilde{\pi}_E - \tilde{\pi}_F)\|_{pk} \\ &\leq k \|f\|_{Q_A} (C(pk) S^{\frac{1}{\alpha(pk)}} h^{\frac{1}{2}})^{k-1} C(pk) h^{\frac{1}{2}} \left(\sum \lambda_s |(\pi_E - \pi_F)(u_s)|^{\alpha(pk)} \right)^{\frac{1}{\alpha(pk)}}. \end{aligned}$$

Then one proceeds as in the preceding proposition, replacing E and F by the terms of an increasing sequence of $\mathcal{F}(H)$ whose union is dense in H and whose first term is E . \square

Remark 3.13. This result holds with the class defined by Remark 3.7, with $C_k(f)$ instead of $\|f\|_{Q_A}$.

A consequence of Lemma 3.8 is the following result, which partly generalizes Proposition 8.7 of [AJN]:

Corollary 3.14. Let $h > 0$ and $p \in [1, +\infty[$. The function $Q_A^{\frac{1}{2}}$ admits a stochastic extension in $L^p(B, \mu_{B,h})$.

Proof. As in the proof of Proposition 3.10, one introduces an increasing sequence (E_n) of $\mathcal{F}(H)$. One denotes by S_{mn} an orthogonal supplement of E_m in E_n if $m \leq n$. Lemma 3.8 then implies the inequality

$$\|Q_A^{\frac{1}{2}} \circ \tilde{\pi}_{S_{mn}}\|_{L^p(B, \mu_{B,h})} \leq C(p) \left(\sum_0^\infty \lambda_j |\pi_{S_{mn}}(u_j)|^{\alpha(p)} \right)^{1/\alpha(p)} h^{\frac{1}{2}},$$

which proves that $(Q_A^{\frac{1}{2}} \circ \tilde{\pi}_{E_n})_n$ is a Cauchy sequence in $L^p(B, \mu_{B,h})$. \square

We finally may state the following result, which enables us to use another Wiener space associated with H than the space B initially chosen.

Proposition 3.15. *Let A be a linear, selfadjoint, nonnegative, trace class application in a Hilbert space H . There exists a measurable norm (see [K] (Def.4.4) or [G-1]), $\|\cdot\|_{A,n}$ on H , and hence a completion B_A of H with respect to this norm, such that the following property is satisfied: if f belongs to the class $S(Q_A)$, then f is uniformly continuous on H with respect to the norm $\|\cdot\|_{A,n}$.*

The function f admits a uniformly continuous extension f_A on B_A and the stochastic extension \tilde{f} of f given by Proposition 3.10 is equal to f_A $\mu_{B,h}$ - a.e.

Proof. If A is an injection, one sets $\|x\|_{A,n} = \langle Ax, x \rangle^{1/2} = Q_A(x)^{1/2}$. If not, if $(e_n)_n$ is an orthonormal basis of $\text{Ker}(A)$, one can add to A the operator C defined, for example, by $Cx = \sum_n e^{-n} \langle x, e_n \rangle e_n$. The operator $A + C$ is selfadjoint, nonnegative, trace class and it is an injection. One then sets $\|x\|_{A,n} = \langle (A + C)x, x \rangle^{1/2} = Q_{A+C}(x)^{1/2}$. It follows from Theorem 3 in [G-1] that $\|\cdot\|_{A,n}$ is a measurable seminorm. It is a norm since A (or $A + C$) is injective. Taylor's formula gives, in both cases, the inequality

$$|f(y) - f(x)| \leq \|f\|_{Q_A} Q_A(y - x)^{1/2} \leq \|f\|_{Q_A} \|x - y\|_{A,n},$$

which in turn implies the uniform continuity. The topological extension f_A of f is then uniformly continuous on B_A . According to Theorem 6.3 (Chap 1 [K]), f_A and \tilde{f} coincide almost everywhere. \square

3.3 Scalar products and products of scalar products

Lemma 3.16. *For all $a \in H$, the function $\varphi_a : H \rightarrow \mathbb{R}, x \mapsto \langle x, a \rangle$ admits the function ℓ_a as a stochastic extension in $L^p(B, \mu_{B,h})$, for all $p \in [1, \infty[$ and all $h > 0$. Moreover, for all $E \in \mathcal{F}(H)$,*

$$\|\varphi_a \circ \widetilde{\pi_E} - \ell_a\|_{L^p(B, \mu_{B,h})} = K(p)h^{\frac{1}{2}} |\pi_E(a) - a|. \quad (22)$$

Proof. Let $(E_j)_j$ be an increasing sequence of $\mathcal{F}(H)$ such that $\bigcup E_j = H$. Since $\varphi_a \circ \widetilde{\pi_{E_j}} = \ell_{\pi_{E_j}(a)}$, according to (5), one obtains, using (7), that, for a finite p

$$\|\varphi_a \circ \widetilde{\pi_{E_j}} - \ell_a\|_{L^p(B, \mu_{B,h})} = \|\ell_{\pi_{E_j}(a)} - \ell_a\|_{L^p(B, \mu_{B,h})} = K(p)h^{\frac{1}{2}} |\pi_{E_j}(a) - a|.$$

This proves the convergence and the result. \square

We now study the products of scalar products. Let a_1, \dots, a_n be vectors of H . Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multiindex such that $\alpha_i > 0$ for every i . One defines the function a^α on H by

$$a^\alpha(x) = \prod_{i=1}^n \langle a_i, x \rangle^{\alpha_i}.$$

Proposition 3.17. *For $h > 0$ and $p \in [1, +\infty[$, the function a^α admits the function $\prod_{i=1}^n \ell_{a_i}^{\alpha_i}$ as a stochastic extension in $L^p(B, \mu_{B,h})$. Moreover, for all $E \in \mathcal{F}(H)$,*

$$\left\| a^\alpha \circ \widetilde{\pi_E} - \prod_{i=1}^n \ell_{a_i}^{\alpha_i} \right\|_{L^p(B, \mu_{B,h})} \leq K(p|\alpha|)^{|\alpha|} h^{|\alpha|/2} \left(\max_{1 \leq i \leq n} |a_i| \right)^{|\alpha|-1} \sum_{i=1}^n \alpha_i |\pi_E(a_i) - a_i|.$$

Proof. One obtains the inequality by applying Lemma 6.1 stated in the appendix to the $|\alpha|$ functions appearing in the products $a^\alpha \circ \widetilde{\pi_E}$ and $\prod_{i=1}^n \ell_{a_i}^{\alpha_i}$ and by remarking that

$$\|\ell_{\pi_E(a_i)}\|_{p|\alpha|} = K(p|\alpha|)h^{\frac{1}{2}} |\pi_E(a_i)| \leq K(p|\alpha|)h^{\frac{1}{2}} |a_i|.$$

It then remains to replace E by an increasing sequence of $\mathcal{F}(H^2)$ such that $\bigcup E_j = H^2$ to obtain the stochastic extension. \square

Besides, one can define a quadratic form associated with such a product thanks to the following result:

Proposition 3.18. *Let $a_1, \dots, a_n, b_1, \dots, b_p$ belong to H , let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p$ be positive integers and set $m = \max(\sum_1^n \alpha_j, \sum_1^p \beta_j) = \max(|\alpha|, |\beta|)$. The function $\tilde{F} : (x, \xi) \mapsto \prod_{i=1}^n \ell_{a_i}^{\alpha_i}(x) \prod_{i=1}^p \ell_{b_i}^{\beta_i}(\xi)$ has a finite*

norm N_m defined by (16). More precisely,

$$\begin{aligned} N_m(\tilde{F}) &= \sup_{Y \in H^2} \frac{\|\tilde{F}(\cdot + Y)\|_{L^1(B^2, \mu_{B^2, \frac{h}{2}})}}{(1 + |Y|)^m} \\ &\leq \max(1, \sqrt{\frac{h}{2}})^{|\alpha|+|\beta|} \prod_1^n |a_j|^{\alpha_j} \prod_1^p |b_i|^{\beta_i} \times \prod_1^n \left(\int_{\mathbb{R}} (1 + |v|)^{n\alpha_j} d\mu_{\mathbb{R},1}(v) \right)^{\frac{1}{n}} \prod_1^p \left(\int_{\mathbb{R}} (1 + |v|)^{p\beta_i} d\mu_{\mathbb{R},1}(v) \right)^{\frac{1}{p}} \end{aligned} \quad (23)$$

Proof. By the change of variables formula (11) one obtains

$$\int_{B^2} |\tilde{F}(X+Y)| d\mu_{B^2, \frac{h}{2}}(X) \leq e^{-\frac{1}{h}|y|^2} \int_B \prod_{j=1}^n |\ell_{a_j}(x)|^{\alpha_j} e^{\frac{2}{h}\ell_y(x)} d\mu_{B, \frac{h}{2}}(x) e^{-\frac{1}{h}|\eta|^2} \int_B \prod_{j=1}^p |\ell_{b_j}(\xi)|^{\beta_j} e^{\frac{2}{h}\ell_\eta(\xi)} d\mu_{B, \frac{h}{2}}(\xi).$$

Hölder's inequality yields

$$A := e^{-\frac{1}{h}|y|^2} \int_B \prod_{j=1}^n |\ell_{a_j}(x)|^{\alpha_j} e^{\frac{2}{h}\ell_y(x)} d\mu_{B, \frac{h}{2}}(x) \leq e^{-\frac{1}{h}|y|^2} \prod_{j=1}^n \left(\int_B |\ell_{a_j}(x)|^{n\alpha_j} e^{\frac{2}{h}\ell_y(x)} d\mu_{B, \frac{h}{2}}(x) \right)^{1/n}.$$

According to (9),

$$A \leq e^{-\frac{1}{h}|y|^2} \prod_{j=1}^n \left(e^{|y|^2/h} \int_{\mathbb{R}} |\sqrt{\frac{h}{2}}|a_j|v + \langle y, a_j \rangle|^{n\alpha_j} d\mu_{\mathbb{R},1}(v) \right)^{1/n}.$$

One can factor $|a_j|$ and, remarking that $\sqrt{\frac{h}{2}}|v| + |y|$ is smaller than $\max(1, \sqrt{\frac{h}{2}})(1 + |v|)(1 + |y|)$, one gets

$$A \leq \max(1, \sqrt{\frac{h}{2}})^{|\alpha|} \prod_1^n |a_j|^{\alpha_j} (1 + |y|)^{|\alpha|} \prod_1^n \left(\int_{\mathbb{R}} (1 + |v|)^{n\alpha_j} d\mu_{\mathbb{R},1}(v) \right)^{1/n}.$$

One treats the other factor similarly, which gives the desired result. \square

3.4 Stochastic extension in an integral

Lemma 3.19. *Let $f : [0, 1] \rightarrow \mathbb{R}$ and $g : [0, 1] \times B^2 \rightarrow \mathbb{R}$ be a measurable functions. Let $q \in [1, +\infty[$ and let $h > 0$. If*

$$\int_0^1 |f(s)| \left(\int_{B^2} |g(s, Y)|^q d\mu_{B^2, h}(Y) \right)^{1/q} ds < \infty,$$

then $Y \mapsto \int_0^1 f(s)g(s, Y) ds$ belongs to $L^q(B^2, \mu_{B^2, h})$ and

$$\left\| \int_0^1 f(s)g(s, \cdot) ds \right\|_{L^q(B^2, \mu_{B^2, h})} \leq \int_0^1 |f(s)| \left(\int_{B^2} |g(s, Y)|^q d\mu_{B^2, h}(Y) \right)^{1/q} ds.$$

Proof. When $q = 1$, the result is straightforward. If $q > 1$, one introduces a function of $L^{q'}(B^2, \mu_{B^2, h})$ (where q' is the conjugate exponent of q) and one proves that the integral belongs to the dual space of $L^{q'}$. \square

Proposition 3.20. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function, let G be in $S_1(\mathcal{B}, \varepsilon)$, with ε summable and let us fix $X \in H^2$. For all $q \in [1, +\infty[$ and all $h \in]0, 1]$, the function*

$$Y \mapsto \int_0^1 f(s)G(X + sY) ds,$$

defined on H^2 , admits, as a stochastic extension in $L^q(B^2, \mu_{B^2, h})$, the function

$$Y \mapsto \int_0^1 f(s)\tilde{G}(X + sY) ds,$$

defined on B^2 , where \tilde{G} is the stochastic extension of G for all $L^r(B^2, \mu_{B^2, h}), (r, h) \in [1, +\infty[\times]0, 1]$. Moreover, if $E \in \mathcal{F}(H^2)$, one has the inequality

$$\begin{aligned} & \left\| \int_0^1 f(s) \left(G(X + s\tilde{\pi}_E(\cdot)) - \tilde{G}(X + s\cdot) \right) ds \right\|_{L^q(B^2, \mu_{B^2, h})} \\ & \leq \|G\|_{1, \varepsilon} \left(\int_0^1 |f(s)| ds \right) \left(\sqrt{2 \sum_{\Gamma} \varepsilon_j^2} |X - \pi_E(X)| + \sqrt{2h(q+2)} C e^{\frac{|X|^2}{2h}} \sum_0^\infty \varepsilon_j (|u_j - \pi_E(u_j)| + |v_j - \pi_E(v_j)|) \right), \end{aligned}$$

where the constant C does not depend on the parameters.

Proof. One checks that all the functions $(s, Y) \mapsto X + sY, X + s\tilde{\pi}_E(Y)$ are measurable. Set $U_E = \int_0^1 f(s)(G(X + s\tilde{\pi}_E(\cdot)) - \tilde{G}(X + s\cdot)) ds$. Using the lemma above, one sees that

$$\begin{aligned} \|U_E\|_{L^q(B^2, \mu_{B^2, h})} & \leq \int_0^1 |f(s)| \|G(X + s\tilde{\pi}_E(\cdot)) - G(\tilde{\pi}_E(X + s\cdot))\|_{L^q(B^2, \mu_{B^2, h})} ds \\ & \quad + \int_0^1 |f(s)| \|G(\tilde{\pi}_E(X + s\cdot)) - \tilde{G}(X + s\cdot)\|_{L^q(B^2, \mu_{B^2, h})} ds. \end{aligned}$$

Formula (15) proves that the first term is smaller than

$$\int_{B^2} |G(X + s\tilde{\pi}_E(Y)) - G(\tilde{\pi}_E(X + sY))|^q d\mu_{B^2, h} \leq \int_{B^2} \left(\|G\|_{1, \varepsilon} \sqrt{2 \sum_{\Gamma} \varepsilon_j^2} |X - \pi_E(X)| \right)^q d\mu_{B^2, h}.$$

For the second term, successive change of variables give

$$\begin{aligned} \int_{B^2} |G(\tilde{\pi}_E(X + sY)) - \tilde{G}(X + sY)|^q d\mu_{B^2, h}(Y) & = \int_{B^2} |G(\tilde{\pi}_E(X + Z)) - \tilde{G}(X + Z)|^q d\mu_{B^2, s^2 h}(Z) \\ & = \int_{B^2} |G(\tilde{\pi}_E(Z)) - \tilde{G}(Z)|^q e^{-\frac{|X|^2}{2s^2 h}} e^{\frac{1}{s^2 h} \ell_X(Z)} d\mu_{B^2, s^2 h}(Z). \end{aligned}$$

One then applies Hölder's inequality to the last term, raising $|G(\tilde{\pi}_E(Z)) - \tilde{G}(Z)|^q$ to the power q'/q with $q' = q + \frac{1}{s^2}$. This gives

$$\|G(\tilde{\pi}_E(X + s\cdot)) - \tilde{G}(X + s\cdot)\|_{L^q(B^2, \mu_{B^2, h})} \leq e^{\frac{|X|^2}{2h}} \|G \circ \tilde{\pi}_E - \tilde{G}\|_{L^{q+\frac{1}{s^2}}(B^2, \mu_{B^2, s^2 h})}.$$

Using (17), one obtains

$$\begin{aligned} & \int_0^1 |f(s)| \|G(\tilde{\pi}_E(X + s\cdot)) - \tilde{G}(X + s\cdot)\|_{L^q(B^2, \mu_{B^2, h})} ds \\ & \leq \int_0^1 |f(s)| e^{\frac{|X|^2}{2h}} \|G\|_{1, \varepsilon} \sqrt{2hs^2} \left(\pi^{-\frac{1}{2}} \Gamma \left(\frac{q+s^{-2}+1}{2} \right) \right)^{\frac{1}{q+s^{-2}}} \sum_0^\infty \varepsilon_j (|u_j - \pi_E(u_j)| + |v_j - \pi_E(v_j)|) ds. \end{aligned}$$

For large values of $|z|$ and $|\arg(z)| < \pi$, one has (see [MOS] p 12 for example)

$$\Gamma(z) = z^{-\frac{1}{2}} e^{z(\ln(z)-1)} \sqrt{2\pi} (1 + O(z^{-1})).$$

It follows that, for a constant C which is independent of the parameters,

$$\left(\pi^{-\frac{1}{2}} \Gamma \left(\frac{q+s^{-2}+1}{2} \right) \right)^{\frac{1}{q+s^{-2}}} \leq C(q+s^{-2}+1)^{\frac{1}{2}},$$

which gives the estimate for the second term. \square

By differentiating under the integral sign one can obtain a weaker result, implying the existence of a stochastic extension, but not its precise form:

Proposition 3.21. *Let $G \in S_m(\mathcal{B}, \varepsilon)$ and let $f \in L^1([0, 1])$, with values in \mathbb{R} . Let $X \in H^2$. For all $Y \in H^2$, one sets*

$$T_X G(Y) = \int_0^1 f(s) G(X + sY) ds.$$

Then $T_X G \in S_m(\mathcal{B}, \varepsilon)$ and

$$\|T_X G\|_{m, \varepsilon} \leq \int_0^1 |f(s)| ds \|G\|_{m, \varepsilon}.$$

Corollary 3.22. *Let f be a continuous function on $[0, 1]$, with values in \mathbb{R} . Let G be in $S_1(\mathcal{B}, \varepsilon)$, with ε summable. Let a_1, \dots, a_n and X belong to H^2 and let (p, h) be in $[1, +\infty[\times \mathbb{R}^{+*}$. The function*

$$Y \in H^2 \mapsto \left(\prod_{i=1}^k \langle a_i, Y \rangle \right) \int_0^1 f(s) G(X + sY) \, ds$$

admits as a stochastic extension in $L^p(B^2, \mu_{B^2, h})$ the function

$$Y \in B^2 \mapsto \left(\prod_{i=1}^k \ell_{a_i}(Y) \right) \int_0^1 f(s) \tilde{G}(X + sY) \, ds,$$

where \tilde{G} is the stochastic extension of G valid for all $h' \leq h_0 := 2h$ and all finite p . Moreover, there exists a constant K depending on p, k, h but not on the a_i, G, X, f, E or ε such that, for all $E \in \mathcal{F}(H^2)$,

$$\begin{aligned} & \left\| \left(\prod_{i=1}^k \langle a_i, \tilde{\pi}_E(Y) \rangle \right) \int_0^1 f(s) G(X + s\tilde{\pi}_E(Y)) \, ds - \left(\prod_{i=1}^k \ell_{a_i}(Y) \right) \int_0^1 f(s) \tilde{G}(X + sY) \, ds \right\|_{L^p(B^2, \mu_{B^2, h})} \\ & \leq K \int_0^1 |f(s)| \, ds \|G\|_{1, \varepsilon} A^{k-1} \times \\ & \quad \left(\sum_{i=1}^k |\pi_E(a_i) - a_i| + A \sqrt{\sum_{j=0}^{\infty} \varepsilon_j^2 |\pi_E(X) - X| + A e^{|X|^2/2h} \sum_{j=0}^{\infty} \varepsilon_j (|\pi_E(u_j) - u_j| + |\pi_E(v_j) - v_j|)} \right) \end{aligned}$$

where $A = \max_{1 \leq i \leq k} (|a_i|)$.

Proof. One uses (41) to establish that

$$\left\| \left(\prod_{i=1}^k \langle a_i, \tilde{\pi}_E(Y) \rangle \right) \int_0^1 f(s) G(X + s\tilde{\pi}_E(Y)) \, ds - \left(\prod_{i=1}^k \ell_{a_i}(Y) \right) \int_0^1 f(s) \tilde{G}(X + sY) \, ds \right\|_{L^p(B^2, \mu_{B^2, h})}$$

is smaller than

$$\begin{aligned} & \prod_{i=1}^k \|\ell_{a_i}\| \times \left\| \int_0^1 f(s) G(X + s\tilde{\pi}_E(Y)) \, ds - \int_0^1 f(s) \tilde{G}(X + sY) \, ds \right\| \\ & + \sum_{i=1}^k \left(\prod_{j=1}^{i-1} \|\ell_{\pi_E(a_j)}\| \prod_{j=i+1}^k \|\ell_{a_j}\| \right) \|\ell_{\pi_E(a_i)} - \ell_{a_i}\| \times \left\| \int_0^1 f(s) G(X + s\tilde{\pi}_E(Y)) \, ds \right\|, \end{aligned}$$

the norm in the second term being the $L^{p(k+1)}(B^2, \mu_{B^2, h})$ -norm. But $\|\ell_a\| = K(p(k+1))h^{\frac{1}{2}}|a|$, according to (7) and (8). An upper bound is, consequently,

$$\begin{aligned} & (K(p(k+1))h^{\frac{1}{2}})^k \left(\prod_{i=1}^k |a_j| \right) \left\| \int_0^1 f(s) G(X + s\tilde{\pi}_E(Y)) \, ds - \int_0^1 f(s) \tilde{G}(X + sY) \, ds \right\| \\ & + (K(p(k+1))h^{\frac{1}{2}})^k \sum_{i=1}^k |\pi_E(a_i) - a_i| \left(\prod_{1 \leq j \leq k, j \neq i} |a_j| \right) \times \left\| \int_0^1 f(s) G(X + s\tilde{\pi}_E(Y)) \, ds \right\| \end{aligned}$$

One concludes by remarking that $|G|$ is smaller than $\|G\|_{1, \varepsilon}$, (thanks to Proposition 3.20) and that the $|a_i|$ are smaller than A . \square

4 Taylor expansions

4.1 Differentiability of the symbols in $S_m(\mathcal{B}, \varepsilon)$

In this subsection, the sequence ε is supposed to be square summable in most results.

The following straightforward lemma lists useful properties of the S_m classes:

Lemma 4.1. Let $F \in S_m(\mathcal{B}, \varepsilon)$, with ε square summable.

- If $m \geq 1$, for all $i \in \Gamma$, $\partial_{u_i} F$ and $\partial_{v_i} F$ belong to $S_{m-1}(\mathcal{B}, \varepsilon)$ and $\|\partial_{u_i} F\|_{m-1, \varepsilon} \leq \varepsilon_i \|F\|_{m, \varepsilon}$, $\|\partial_{v_i} F\|_{m-1, \varepsilon} \leq \varepsilon_i \|F\|_{m, \varepsilon}$. More generally, if $m \geq k \geq 1$ and if α, β are two multi-indices of depth k (such that $\max_{j \in \Gamma} (\alpha_j, \beta_j) \leq k$), then $\partial_u^\alpha \partial_v^\beta F \in S_{m-k}(\mathcal{B}, \varepsilon)$ and

$$\|\partial_u^\alpha \partial_v^\beta F\|_{m-k, \varepsilon} \leq \|F\|_{m, \varepsilon} \prod_{j \in \Gamma} \varepsilon_j^{\alpha_j + \beta_j}.$$

- If $m \geq 2$, one defines $\Delta_{\mathcal{B}}$ by

$$\Delta_{\mathcal{B}} F = \left(\sum_{j \in \Gamma} \left(\frac{\partial}{\partial u_j} \right)^2 + \left(\frac{\partial}{\partial v_j} \right)^2 \right) F.$$

It is well defined and $\Delta_{\mathcal{B}} F \in S_{m-2}(\mathcal{B}, \varepsilon)$, with $\|\Delta_{\mathcal{B}} F\|_{m-2, \varepsilon} \leq 2 \sum_j \varepsilon_j^2 \|F\|_{m, \varepsilon}$.

- If $G \in S_m(\mathcal{B}, \delta)$ with δ square summable too, then $FG \in S_m(\mathcal{B}, \varepsilon + \delta)$ with $\|FG\|_{m, \varepsilon + \delta} \leq \|F\|_{m, \varepsilon} \|G\|_{m, \delta}$.

One can prove that, under certain conditions, the Laplace operator does not depend on the chosen basis (see Remark 4.8 below).

Proposition 4.2. If $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$ and ε square summable, then F is Fréchet differentiable on H^2 and

$$DF(X) \cdot Y = \sum_{j \in \Gamma} \langle Y, u_j \rangle \frac{\partial F}{\partial u_j}(X) + \langle Y, v_j \rangle \frac{\partial F}{\partial v_j}(X).$$

Moreover, for all X and Y in H^2 ,

$$|F(X + Y) - F(X) - DF(X) \cdot Y| \leq \|F\|_{m, \varepsilon} \sum_{j \in \Gamma} \varepsilon_j^2 (1 + 2\sqrt{2}) |Y|^2.$$

Proof. Let $X, Y \in H^2$. Suppose that Γ is enumerated. Let P_N be the orthogonal projection onto $\text{Vect}(u_i, v_i, i \leq N)$ if $N \geq 0$, $P_{-1} = 0$ and $P_{N, \frac{1}{2}}$ the orthogonal projection onto $\text{Vect}(u_i, v_j, i \leq N+1, j \leq N)$. By approaching $P(X + Y)$ by $P(X + P_N(Y))$ one obtains

$$F(X + P_N(Y)) - F(X) = \sum_{j=0}^N F(X + P_j(Y)) - F(X + P_{j-1, \frac{1}{2}}(Y)) + F(X + P_{j-1, \frac{1}{2}}(Y)) - F(X + P_{j-1}(Y)).$$

Taylor's formula gives, for example for the part of the j -th term concerned with v_j ,

$$\begin{aligned} & F(X + \langle Y, v_j \rangle v_j + P_{j-1, \frac{1}{2}}(Y)) - F(X + P_{j-1, \frac{1}{2}}(Y)) \\ &= \langle Y, v_j \rangle \frac{\partial F}{\partial v_j}(X + P_{j-1, \frac{1}{2}}(Y)) \\ &+ \langle Y, v_j \rangle^2 \int_0^1 (1-s) \frac{\partial^2 F}{\partial v_j^2}(X + P_{j-1, \frac{1}{2}}(Y) + s \langle Y, v_j \rangle v_j) ds \\ &= \langle Y, v_j \rangle \frac{\partial F}{\partial v_j}(X) \\ &+ \langle Y, v_j \rangle \left(\frac{\partial F}{\partial v_j}(X + P_{j-1, \frac{1}{2}}(Y)) - \frac{\partial F}{\partial v_j}(X) \right) \\ &+ \langle Y, v_j \rangle^2 \int_0^1 (1-s) \frac{\partial^2 F}{\partial v_j^2}(X + P_{j-1, \frac{1}{2}}(Y) + s \langle Y, v_j \rangle v_j) ds. \end{aligned}$$

The first term gives the expression of the differential and it is the general term of a convergent series (apply Cauchy-Schwarz inequality). Since $\frac{\partial F}{\partial v_j}$ is in $S_{m-1}(\mathcal{B}, \varepsilon)$ with $\left\| \frac{\partial F}{\partial v_j} \right\|_{m-1, \varepsilon} \leq \varepsilon_j \|F\|_{m, \varepsilon}$, one can use (15) to treat the second term. It then yields a convergent series too, its sum being smaller than $\text{Cste} \cdot |Y|^2$. The integral term can be estimated thanks to the estimates on the second derivatives and the sum of the corresponding terms is of order 2 in $|Y|$. Since F and its derivatives are bounded by $\|F\|_{m, \varepsilon}$ and powers of ε independently on X and Y , the rest can be bounded as is asserted in the theorem, with a constant C independent of $X, Y, \|F\|_{m, \varepsilon}$ and ε . One can take $C = (1 + 2\sqrt{2})$. \square

Remark. Since there are infinitely many terms, we need a precise bound for the rest in Taylor's formula, which explains the loss of one order of differentiability.

Deriving term by term and using the continuity of the extension operator \mathcal{P} (Corollary 3.3) gives the following results:

Proposition 4.3. *Let $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$, ε square summable. Then, for all $Y \in H^2$, $X \mapsto DF(X) \cdot Y$ is in $S_{m-1}(\mathcal{B}, \varepsilon)$, with $\|X \mapsto DF(X) \cdot Y\|_{m-1, \varepsilon} \leq 2\|F\|_{m, \varepsilon}|Y|\sqrt{\sum_{j \in \Gamma} \varepsilon_j^2}$.*

Corollary 4.4. *Let $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$ and ε summable. The application $X \mapsto DF(X) \cdot Y$ from H in \mathbb{R} admits a stochastic extension in $L^p(B^2, \mu_{B^2, t})$, which is the application*

$$\sum_{\Gamma} \langle Y, u_j \rangle \mathcal{P} \left(\frac{\partial F}{\partial u_j} \right) + \langle Y, v_j \rangle \mathcal{P} \left(\frac{\partial F}{\partial v_j} \right). \quad (24)$$

Here, the summability of ε is needed to ensure the existence of the stochastic extension.

Definition 4.5. *Let $F \in S_m(\mathcal{B}, \varepsilon)$ with ε square summable. For $k \in \{1, \dots, m\}$ and $X \in H^2$, one defines a k -linear symmetric continuous form $\Phi_k(X)$ on $(H^2)^k$ setting:*

$$\begin{aligned} \forall (Y_1, \dots, Y_k) \in (H^2)^k, \\ \Phi_k(X)(Y_1, \dots, Y_k) = \sum_{\substack{(j_1, \dots, j_k) \in \Gamma^k, \\ (\delta_1, \dots, \delta_k) \in \{0, 1\}^k}} \left(\prod_{s=1}^k \langle Y_s, w_{j_s}^{\delta_s} \rangle \right) \frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}}(X), \end{aligned}$$

with $w_j^0 = u_j, w_j^1 = v_j$. Moreover

$$\forall X, Y_1, \dots, Y_k \in (H^2)^{k+1}, |\Phi_k(X)(Y_1, \dots, Y_k)| \leq 2^k \|F\|_{m, \varepsilon} \prod_{s=1}^k |Y_s| \left(\sum_{\Gamma} \varepsilon_j^2 \right)^{\frac{k}{2}}. \quad (25)$$

From now on, for the sake of brevity, we shall write $J \in \Gamma^k, \delta \in \{0, 1\}^k$ instead of $(j_1, \dots, j_k) \in \Gamma^k, (\delta_1, \dots, \delta_k) \in \{0, 1\}^k$.

Proposition 4.6. *Let $F \in S_m(\mathcal{B}, M, \varepsilon)$ with ε square summable. Then F is C^{m-1} on H^2 and, for all $k \in \{1, \dots, m-1\}$ and all $X \in H^2$,*

$$D^k F(X) = \Phi_k(X).$$

The inequality (25) is satisfied. Finally, for $0 \leq k \leq m-2$, one has

$$||| D^k F(X+Z) - D^k F(X) - D^{k+1} F(X)(\cdot, Z) ||| \leq 2^k \|F\|_{m, \varepsilon} \left(\sum_{\Gamma} \varepsilon_j^2 \right)^{(k+2)/2} (1 + 2\sqrt{2}) |Z|^2,$$

where the norm is the norm of k -linear continuous applications on H^2 .

Proof. Propositions 4.2 and 4.3 give the result for $m=2$, except for the fact that F is C^1 . This can be proved by applying (15) to the partial derivatives of F . For a general m , one uses induction. \square

This allows to state Taylor's formula to the order k for $F \in S_m(\mathcal{B}, \varepsilon)$, with ε square summable and $m \geq k+1$. For $X, Y \in H^2$,

$$\begin{aligned} F(X+Y) &= F(X) + \sum_{i=1}^{k-1} \frac{1}{i!} D^i F(X) \cdot Y^i + \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} D^k F(X+sY) \cdot Y^k ds \\ &= F(X) + \sum_{i=1}^{k-1} \frac{1}{i!} D^i F(X) \cdot Y^i \\ &\quad + \sum_{J \in \Gamma^k, \delta \in \{0, 1\}^k} \left(\prod_{r=1}^k \langle w_{j_r}^{\delta_r}, Y \rangle \right) \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} \frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}}(X+sY) ds, \end{aligned} \quad (26)$$

exchanging the sums to get the last equality.

One part of the following subsection 4.2 proves the existence of stochastic extensions for each of the terms appearing here, the polynomial terms as well as the rest, under the assumption that ε is summable. Note that these extensions are series indexed by Γ .

One can finally state the following result, which allows us to construct another completion B_A of H in the case when ε is summable.

Proposition 4.7. *Let ε be a summable sequence such that $\varepsilon_j > 0$ for all $j \in \Gamma$. One defines a symmetric, definite positive and trace class operator A by setting*

$$\forall X \in B^2, \quad AX = \sum_{j \in \Gamma} \varepsilon_j \langle X, u_j \rangle u_j + \varepsilon_j \langle X, v_j \rangle v_j.$$

Set $\|X\|_A = \langle AX, X \rangle^{1/2}$. Then $\|\cdot\|_A$ is a measurable norm on H , in the sense of [K] (Def.4.4) or [G-1]. One denotes by B_A the completion of H for this norm.

If $F \in S_m(\mathcal{B}, \varepsilon)$ for $m \geq 2$, then F is uniformly continuous on H^2 with respect to the norm $\|\cdot\|_A$. The function F admits a uniformly continuous extension F_A on B_A and the stochastic extension \tilde{F} of F given by Proposition 3.1 is equal to F_A $\mu_{B,h}$ - a.e.

Proof. It follows from Theorem 3 in [G-1] that $\|\cdot\|_A$ is a measurable norm, since A is injective. Since $m \geq 2$, F is C^1 on H . Taylor's formula with an integral rest and Definition 4.5 allow us to write the inequality

$$\begin{aligned} |F(X) - F(Y)| &\leq \int_0^1 \sum_{j \in \{0,1\}, \delta \in \Gamma} \left| \frac{\partial F}{\partial w_j^\delta}(X + t(Y - X)) \langle Y - X, w_j^\delta \rangle \right| dt \\ &\leq \sum_{j \in \Gamma} \|F\|_{m,\varepsilon} \varepsilon_j^{1/2} \varepsilon_j^{1/2} (|\langle Y - X, u_j \rangle| + |\langle Y - X, v_j \rangle|) \\ &\leq \|F\|_{m,\varepsilon} \sqrt{2} \left(\sum \varepsilon_j \right)^{1/2} \|X - Y\|_A, \end{aligned}$$

thanks to Cauchy-Schwarz inequality. This proves that F is uniformly continuous on H^2 and therefore admits an extension F_A , which is uniformly continuous on B_A . According to Theorem 6.3 (Chap 1 [K]), F_A and \tilde{F} coincide almost everywhere. \square

Remark 4.8. *If $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 3$ and ε summable, one can define ΔF more intrinsically. Indeed, one can state an inequality more precise than (25). For $k \leq 3$ one gets*

$$\forall X, Y_1, \dots, Y_k \in (H^2)^{k+1}, \quad |\Phi_k(X)(Y_1, \dots, Y_k)| \leq 2^k \|F\|_{m,\varepsilon} \left(\sum_{\Gamma} \varepsilon_j \right)^{k/2} \prod_{s=1}^k \langle AY_s, Y_s \rangle^{1/2},$$

reasoning as in the proof of Proposition 4.7. The function F is C^2 since $m \geq 3$ and the inequality, for $k = 2$, ensures the existence of a self adjoint, trace class operator M_x satisfying

$$\forall U, V, X \in H^2, \quad d^2 F(X) \cdot (U, V) = \langle M_X U, V \rangle.$$

One then sets $\Delta F(X) = \text{Tr}(M_X)$ and the expression as a sum of partial derivatives does not depend on the chosen orthonormal basis.

One can remark, too, that if ε is summable, if F belongs to $S_m(\mathcal{B}, \varepsilon)$ for all m and if there exists a constant M such that $\|F\|_{m,\varepsilon} \leq M$ for all m , then $F \in S(Q_B)$ with B defined by $B = 4(\sum_{\Gamma} \varepsilon_j)A$, A being as in Proposition 4.7.

4.2 Taylor's formula and stochastic expansions

Contrary to the preceding subsection, where sums like $\sum_{\Gamma} \varepsilon_j \langle u_j, x \rangle$ have been treated by Cauchy-Schwarz inequality, we must suppose here that the sequence ε is summable. The corresponding sums have the form $\sum_{\Gamma} \varepsilon_j \ell_{u_j}$ and, since the functions ℓ_{u_j} have a L^p norm independent of j , Cauchy-Schwarz inequality cannot be applied.

Lemma 4.9. *Let ε be a summable sequence. Let $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$ and let $X \in H^2$. For all $k \leq m$ and all $h > 0$, $p \in [1, +\infty[$, the application $Y \mapsto \Phi_k(X) \cdot Y^k$ from Definition 4.5 admits, as a stochastic expansion in $L^p(B^2, \mu_{B^2, h})$, the application $Y \mapsto \widetilde{\Phi_k(X)} \cdot Y^k$ defined on B^2 by*

$$\forall Y \in B^2, \quad \widetilde{\Phi_k(X)} \cdot Y^k = \sum_{J \in \Gamma^k, \delta \in \{0,1\}^k} \left(\prod_{s=1}^k \ell_{w_{j_s}^{\delta_s}}(Y) \right) \frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}}(X),$$

with $w_j^0 = u_j, w_j^1 = v_j$.

Proof. Let $E \in \mathcal{F}(H)$. To verify that $\widetilde{\Phi_k(X)} \cdot Y^k$ and $Y \mapsto \Phi_k(X) \cdot (\tilde{\pi}_E(Y))^k$ really belong to $L^p(B^2, \mu_{B^2, h})$, one has to find an upper bound for each term

$$\left\| \prod_{s=1}^k \ell_{a_s}(Y) \right\|_{L^p(B^2, \mu_{B^2, h})} \left| \frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}}(X) \right|$$

of the sum, with $a_s = w_{j_s}^{\delta_s}$ ou $\pi_E(w_{j_s}^{\delta_s})$. One then proves, using Proposition 3.17, that

$$\left\| \prod_{s=1}^k < \tilde{\pi}_E(Y), w_{j_s}^{\delta_s} > - \prod_{s=1}^k \ell_{w_{j_s}^{\delta_s}} \right\|_{L^p(B^2, \mu_{B^2, h})} \leq (K(pk)h^{\frac{1}{2}})^k \sum_{s=1}^k |\pi_E(w_{j_s}^{\delta_s}) - w_{j_s}^{\delta_s}|,$$

since the $w_{j_s}^{\delta_s}$ and their projections have norms smaller than 1 Therefore

$$\begin{aligned} & \|\Phi_k(X) \cdot (\tilde{\pi}_E(Y))^k - \widetilde{\Phi_k(X)}(Y, \dots, Y)\|_{L^p(B^2, \mu_{B^2, h})} \\ & \leq \|F\|_{m, \varepsilon} (K(pk)h^{\frac{1}{2}})^k \sum_{J \in \Gamma^k, \delta \in \{0,1\}^k} \prod_{s=1}^k \varepsilon_{j_s} \sum_{s=1}^k |\pi_E(w_{j_s}^{\delta_s}) - w_{j_s}^{\delta_s}|. \end{aligned}$$

One then replaces E by E_n , where (E_n) is an increasing sequence of $\mathcal{F}(H^2)$ whose union is dense in H^2 . Since the terms $|\pi_{E_n}(w_{j_s}^{\delta_s}) - w_{j_s}^{\delta_s}|$ converge to 0 and are smaller than 2, the difference converges to 0 thanks to the dominated convergence Theorem. \square

Proposition 4.10. *Let ε be summable and let $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$. Let $X \in H^2$. For all $k \leq m-1$, all $h > 0$ and $p \in [1, +\infty[$, one can write, in $L^p(B^2, \mu_{B^2, h})$:*

$$\begin{aligned} \tilde{F}(X+Y) &= F(X) + \sum_{i=1}^{k-1} \frac{1}{i!} \sum_{J \in \Gamma^i, \delta \in \{0,1\}^i} \left(\prod_{r=1}^i \ell_{w_{j_r}^{\delta_r}}(Y) \right) \frac{\partial^i F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_i}^{\delta_i}}(X) \\ &+ \sum_{J \in \Gamma^k, \delta \in \{0,1\}^k} \left(\prod_{r=1}^k \ell_{w_{j_r}^{\delta_r}}(Y) \right) \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} \mathcal{P} \left(\frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}} \right) (X + sY) ds. \end{aligned} \quad (27)$$

Proof. Let us denote by $\tilde{\Phi}_i(X) \cdot Y^i$ the i -th term of the sum and by $R_k(X)$ the last one, corresponding to the rest. We have just seen that the polynomial part of the development in (26) has a stochastic extension in $L^p(B^2, \mu_{B^2, h})$. The rest is the sum indexed by $J = (j_1, \dots, j_k) \in \Gamma^k, \delta = (\delta_1, \dots, \delta_k) \in \{0,1\}^k$. One applies the Corollary 3.22, replacing, in the upper bound, $\int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} ds$ by $(k!)^{-1}$, $\|G\|_{1, \varepsilon}$ by $\|F\|_{m, \varepsilon} \prod_{i=1}^k \varepsilon_{j_i}$ and

$A = \max(|w_{j_i}^{\delta_i}|)$, by 1. One finds

$$\begin{aligned}
& \sum_{J \in \Gamma^k, \delta \in \{0,1\}^k} \left\| \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} \left(\frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}} (X + s\tilde{\pi}_E(Y)) \prod_1^k < \tilde{\pi}_E(Y), w_{j_i}^{\delta_i} > \right. \right. \\
& \quad \left. \left. - (\mathcal{P} \frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}})(X + sY) \prod_1^k \ell_{w_{j_i}^{\delta_i}}(Y) \right) ds \right\|_{L^p(B^2, \mu_{B^2, h})} \\
& \leq \frac{1}{k!} K \|F\|_{m, \varepsilon} \sum_{J \in \Gamma^k, \delta \in \{0,1\}^k} (\varepsilon_{j_1} \dots \varepsilon_{j_k}) \sum_{i=1}^k |\pi_E(w_{j_i}^{\delta_i}) - w_{j_i}^{\delta_i}| \\
& \quad + \frac{1}{k!} K \|F\|_{m, \varepsilon} \sum_{J \in \Gamma^k, \delta \in \{0,1\}^k} (\varepsilon_{j_1} \dots \varepsilon_{j_k}) \times \\
& \quad \left(\sqrt{\sum_j \varepsilon_j^2} |\pi_E(X) - X| + e^{|X|^2/2h} \sum_0^\infty \varepsilon_j (|\pi_E(u_j) - u_j| + |\pi_E(v_j) - v_j|) \right).
\end{aligned}$$

If one replaces E by E_n from an increasing sequence of $\mathcal{F}(H^2)$ whose union is dense in H^2 , this converges to 0 when n converges to infinity. \square

With each term of the extended Taylor expansion (27), one can associate a quadratic form (see [AJN], Definition 1.2) thanks to the following result:

Proposition 4.11. *Let $F \in S_m(\mathcal{B}, \varepsilon)$ with ε summable and $m \geq k+1$, where k is the order of differentiation. Each of the terms of (27) has a N_s norm (cf. (16)), for a well-chosen s . Precisely*

$$N_i\left(\frac{1}{i!} \tilde{\Phi}_i(X) \cdot Y^i\right) \leq \frac{1}{i!} \|F\|_{m, \varepsilon} \left(2 \max(1, \sqrt{\frac{h}{2}}) \sum_\Gamma \varepsilon_j \right)^i \int_{\mathbb{R}} (1 + |v|)^i d\mu_{\mathbb{R}, 1}(v).$$

and

$$N_k(R_k(X)) \leq \frac{1}{k!} \|F\|_{m, \varepsilon} \left(2 \max(1, \sqrt{\frac{h}{2}}) \sum_\Gamma \varepsilon_j \right)^k \int_{\mathbb{R}} (1 + |v|)^k d\mu_{\mathbb{R}, 1}(v).$$

Proof. One uses the computations of Proposition 3.18. Then

$$\left\| \prod_{r=1}^i \ell_{w_{j_r}^{\delta_r}}(\cdot + Y) \right\|_{L^1(B^2, \mu_{B^2, \frac{h}{2}})} \leq (1 + |Y|)^i \max(1, \sqrt{\frac{h}{2}})^i \int_{\mathbb{R}} (1 + |v|)^i d\mu_{\mathbb{R}, 1}(v).$$

Hence

$$\begin{aligned}
& \left\| \sum_{J \in \Gamma^i, \delta \in \{0,1\}^i} \prod_{r=1}^i \ell_{w_{j_r}^{\delta_r}}(\cdot + Y) \frac{\partial^i F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_i}^{\delta_i}}(X) \right\|_{L^1(B^2, \mu_{B^2, \frac{h}{2}})} \\
& \leq \sum_{J \in \Gamma^i, \delta \in \{0,1\}^i} \|F\|_{m, \varepsilon} \varepsilon_{j_1} \dots \varepsilon_{j_i} (1 + |Y|)^i \max(1, \sqrt{\frac{h}{2}})^i \int_{\mathbb{R}} (1 + |v|)^i d\mu_{\mathbb{R}, 1}(v) \\
& \leq \|F\|_{m, \varepsilon} \left(2 \max(1, \sqrt{\frac{h}{2}}) \sum_\Gamma \varepsilon_j \right)^i \int_{\mathbb{R}} (1 + |v|)^i d\mu_{\mathbb{R}, 1}(v) (1 + |Y|)^i.
\end{aligned}$$

It follows that

$$N_i\left(\frac{1}{i!} \tilde{\Phi}_i(X) \cdot Y^i\right) \leq \frac{1}{i!} \|F\|_{m, \varepsilon} \left(2 \max(1, \sqrt{\frac{h}{2}}) \sum_\Gamma \varepsilon_j \right)^i \int_{\mathbb{R}} (1 + |v|)^i d\mu_{\mathbb{R}, 1}(v).$$

We treat the rest in the same way: the sum indexed by $(j_1, \dots, j_k) \in \Gamma^k, (\delta_1, \dots, \delta_k) \in \{0,1\}^k$ contains a product of k terms ℓ and the integral, which is bounded by $\frac{1}{k!} \|F\|_{m, \varepsilon} \varepsilon_{j_1} \dots \varepsilon_{j_k}$. Therefore the rest has a N_k norm bounded like the polynomial terms. \square

5 The heat operator on H

5.1 Definition

The heat operator defined below associates a function defined on a (real, separable, infinite dimensional) Hilbert space, with a function defined on the same Hilbert space. We aim at extending the notion of heat operator, which is classical in the finite dimensional setting. The results proved here are different from the results obtained by ([K], [G-4]), inasmuch as they are concerned with functions initially defined on H (or H^2) and not on B .

Definition 5.1. Let F be a function defined on H , admitting a stochastic extension in $L^p(B, \mu_{B,t})$ for a given $p \in [1, +\infty[$. One defines $H_t F$ on H by

$$(H_t F)(X) = \int_B \tilde{F}(X + Y) d\mu_{B,t}(Y) = \int_B \tilde{F}(Y) e^{-\frac{|X|^2}{2t}} e^{\ell_X/t} d\mu_{B,t}(Y), \quad (28)$$

the second identity coming from (11).

If F is defined on the product H^2 , one replaces H by H^2 and B by B^2 .

Remark 5.2. This definition does not depend on the stochastic extension chosen, nor on the measurable norm and on the completion of H associated with it. Indeed, the fact that a sequence $F \circ \tilde{\pi}_{E_n}$ is a Cauchy sequence in $L^p(B, \mu_{B,h})$ is expressed by integrals on finite dimensional subspaces of H (using (3)) and not at all by integrals on B . Likewise, the integral of (28) does not depend on the integration space B , since it is a limit of integrals on finite dimensional spaces of H .

Proposition 5.3. Let F belong to a class $S(Q_A)$ of Definition 3.6 or to a class $S_m(\mathcal{B}, \varepsilon)$, with ε summable, of Definition 2.4. The semigroup property is verified: for all positive s, t and all X in the Hilbert space,

$$H_t(H_s F)(X) = H_{t+s} F(X).$$

Moreover, one has (according to whether $F \in S(Q_A)$ or $S_m(\mathcal{B}, \varepsilon)$),

$$\forall X \in H^2, |(H_t F)(X)| \leq \|F\|_{m,\varepsilon} \text{ or } \forall X \in H, |(H_t F)(X)| \leq \|F\|_{Q_A}. \quad (29)$$

Proof. We give the proof in the case when $F \in S(Q_A)$. Let B_A be the completion of H with respect to the measurable norm $\|\cdot\|_A$ given by Proposition 3.15. The function F is uniformly continuous on H and extends continuously as a function denoted by F_A , uniformly continuous and bounded on B_A . By Theorem 6.3 (Chap 1) of [K], every stochastic extension of F in $L^p(B_A, \mu_{B_A,h})$ coincides with F_A $\mu_{B_A,h}$ -a.e. One can thus, considering that the heat operator is being defined by integrating on B_A , write that

$$\forall X \in H, H_t F(X) = \int_{B_A} F_A(X + Y) d\mu_{B_A,t}(Y).$$

This formula allows us to define a function, denoted by $H_t F_A$, on B_A . Since F_A is uniformly continuous and bounded on B_A , $H_t F_A$ is uniformly continuous and bounded on B_A too, by [K] (Theorem 4.1 Chap 3). Then $H_t F_A$ is the stochastic extension of its restriction to H , $H_t F$ and

$$\forall X \in H, H_s(H_t F)(X) = \int_{B_A} H_t F_A(X + Y) d\mu_{B_A,s}(Y) = H_{t+s} F_A(X) = H_{t+s} F(X).$$

For $F \in S_m(\mathcal{B}, \varepsilon)$ with ε summable we can reproduce the same demonstration, with H^2 and $\|\cdot\|_A$, B_A from Proposition 4.7.

The inequalities (29) come from the fact that F_A is bounded on B_A like F on H . \square

5.2 The heat operator in the classes $S_m(\mathcal{B}, \varepsilon)$

Proposition 5.4. Let $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$, ε summable. If α, β are depth 1 multiindices (such that $\max(\alpha_j, \beta_j) \leq 1$), then

$$\partial_u^\alpha \partial_v^\beta (H_t F)(X) = H_t (\partial_u^\alpha \partial_v^\beta F)(X).$$

Moreover, for $m \geq 1$, $H_t F \in S_{m-1}(\mathcal{B}, \varepsilon)$, with $\|H_t F\|_{m-1,\varepsilon} \leq \|F\|_{m,\varepsilon}$. The operator H_t is continuous from $S_m(\mathcal{B}, \varepsilon)$, in $S_{m-1}(\mathcal{B}, \varepsilon)$.

Proof. If $m = 1$, the continuity of H_t from $S_1(\mathcal{B}, \varepsilon)$ in $S_0(\mathcal{B}, \varepsilon)$ comes from the inequalities (29). Now suppose that $m \geq 2$ and prove (first) that

$$\frac{\partial}{\partial w}(H_t F)(X) = H_t \left(\frac{\partial}{\partial w} F \right) (X)$$

with $w = u_i$ or v_i and $X \in H^2$. By Taylor's formula

$$F(X + rw) - F(X) = r \frac{\partial F}{\partial w}(X) + r^2 \int_0^1 (1-s) \frac{\partial^2 F}{\partial w^2}(X + rsw) ds. \quad (30)$$

According to Proposition 3.1 and its corollary, F and $\frac{\partial F}{\partial w}$ together with their translated of a vector $Y \in H^2$ admit stochastic extensions in $L^p(B^2, \mu_{B^2, t})$ and $\widetilde{\tau_Y F} = \tau_Y \widetilde{F}$. According to (30), for all $r \in \mathbb{R}^*$, the function $G_r : X \mapsto \int_0^1 (1-s) \frac{\partial^2 F}{\partial w^2}(X + rsw) ds$ admits a stochastic extension in $L^p(B^2, \mu_{B^2, t})$, denoted by $\widetilde{G_r}$.

For all r , $|G_r| \leq \frac{1}{2} \|F\|_{m, \varepsilon} \sup(\varepsilon_i)^2$. Hence, so does $\widetilde{G_r}$ $\mu_{B^2, t}$ -a.s.

Applying (30) in the point $\widetilde{\pi_{E_j}}(X)$ with $X \in B^2$ and taking a limit in $L^p(B^2, \mu_{B^2, t})$, one obtains

$$\tau_{rw} \widetilde{F} - \widetilde{F} = r \mathcal{P} \left(\frac{\partial F}{\partial w} \right) + r^2 \widetilde{G_r}, \quad \text{in } L^p(B^2, \mu_{B^2, t}) \quad (31)$$

One deduces that, for all X of H^2 ,

$$\frac{(H_t F)(X + rw) - (H_t F)(X)}{r} = \left(H_t \frac{\partial F}{\partial w} \right) (X) + r(H_t G_r)(X),$$

and that

$$\left| \frac{(H_t F)(X + rw) - (H_t F)(X)}{r} - \left(H_t \frac{\partial F}{\partial w} \right) (X) \right| \leq |r| \int_{B^2} |\widetilde{G_r}|(X + Y) d\mu_{B^2, t}(Y).$$

The bound on $\widetilde{G_r}$ shows that

$$\lim_{r \rightarrow 0} \frac{(H_t F)(X + rw) - (H_t F)(X)}{r} = \left(H_t \frac{\partial F}{\partial w} \right) (X),$$

which means that $H_t F$ admits order 1 partial derivatives in the (canonical) directions u_i, v_i .

Let α, β be two depth 1 multiindices. Let $w = u_i$ (or v_i) be a coordinate, with respect to which one has not yet differentiated (that is, such that $\alpha_i = 0$ or $\beta_i = 0$). Applying the preceding reasoning to $\partial_u^\alpha \partial_v^\beta F$, we get that

$$\frac{\partial}{\partial w} H_t(\partial_u^\alpha \partial_v^\beta F)(X) = H_t \left(\frac{\partial}{\partial w} \partial_u^\alpha \partial_v^\beta F \right) (X)$$

and an induction on $|\alpha| + |\beta|$ allows us to exchange H_t and differentiations. By (29), one gets that

$$|\partial_u^\alpha \partial_v^\beta H_t(F)(X)| = |H_t(\partial_u^\alpha \partial_v^\beta F)(X)| \leq \|\partial_u^\alpha \partial_v^\beta F\|_{m-1, \varepsilon} \leq \varepsilon^{\alpha+\beta} \|F\|_{m, \varepsilon}.$$

If $m = 2$, the proposition is proved. Otherwise one completes the proof by induction. \square

The Heat operator commutes with the Laplace operator:

Proposition 5.5. *Let ε be summable. The operator $\Delta_{\mathcal{B}}$ is continuous from $S_m(\mathcal{B}, \varepsilon)$ to $S_{m-2}(\mathcal{B}, \varepsilon)$, for $m \geq 2$. Moreover, for $m \geq 3$,*

$$\forall F \in S_m(\mathcal{B}, \varepsilon), \quad \Delta_{\mathcal{B}} H_t F = H_t \Delta_{\mathcal{B}} F \in S_{m-3}(\varepsilon).$$

Proof. One deduces from Lemma 4.1 that

$$\|\Delta_{\mathcal{B}} F\|_{m-2, \varepsilon} \leq 2 \sum_{j \in \Gamma} \varepsilon_j^2 \|F\|_{m, \varepsilon},$$

which proves the continuity of $\Delta_{\mathcal{B}}$.

One still supposes Γ enumerated. For $n \in \mathbb{N}$, set $\Delta_n = \sum_{j \leq n} \frac{\partial^2}{\partial u_j^2} + \frac{\partial^2}{\partial v_j^2}$. One can see that $\Delta_n F$ converges to $\Delta_{\mathcal{B}} F$ in $S_{m-2}(\mathcal{B}, \varepsilon)$. Moreover, one can exchange H_t and the differentiations with respect to u_j, v_j . This fact, and the continuity of the operators, allow us to write

$$H_t \Delta_{\mathcal{B}} F = H_t \lim_{n \rightarrow \infty} \Delta_n F = \lim_{n \rightarrow \infty} H_t \Delta_n F = \lim_{n \rightarrow \infty} \Delta_n H_t F = \Delta_{\mathcal{B}} H_t F,$$

which completes the proof. \square

Let us state a result about commutators. For $Z \in H^2$ and F a function defined on H^2 , denote by $M_Z F$ the function defined by $(M_Z F)(X) = \langle Z, X \rangle F(X)$.

Proposition 5.6. *Let $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$ and ε square summable. For all $i \in \mathbb{N}$, one has*

$$\frac{1}{t} (H_t M_{u_i} - M_{u_i} H_t) F = H_t \frac{\partial F}{\partial u_i}$$

and then

$$\frac{1}{t} [H_t, M_{u_i}] = H_t \frac{\partial}{\partial u_i}$$

on $S_m(\mathcal{B}, \varepsilon)$. The same property holds with v_i .

Proof. Notice that $M_Z F$ admits $\ell_Z \tilde{F}$ as a stochastic extension in $L^p(B^2, \mu_{B^2, t})$ for all $p \in [1, +\infty[$, by Corollary 6.2. According to Theorem 6.2 (chap. 2, par. 6) of [K], for all $X \in H^2$,

$$\frac{\partial H_t F}{\partial u_i}(X) = \frac{1}{t} \int_{B^2} \tilde{F}(X + Y) \ell_{u_i}(Y) d\mu_{B^2, t}(Y).$$

But $\ell_{u_i}(Y) = \ell_{u_i}(Y + X) - \langle u_i, X \rangle$, since $X \in H^2$. Then

$$\frac{\partial H_t F}{\partial u_i}(X) = \frac{1}{t} \int_{B^2} \tilde{F}(X + Y) \ell_{u_i}(Y + X) d\mu_{B^2, t}(Y) - \langle u_i, X \rangle \frac{1}{t} \int_{B^2} \tilde{F}(X + Y) d\mu_{B^2, t}(Y).$$

This is the desired result. \square

We shall use the Taylor expansions and their stochastic extensions to prove a preliminary result before stating the main result of this subsection, Theorem 5.9.

Proposition 5.7. 1. *Let $m \geq 3$. There exists $C_m \in \mathbb{R}^+$ such that, for all $F \in S_m(\mathcal{B}, \varepsilon)$,*

$$\|H_t F - F\|_{m-3, \varepsilon} \leq C_m \|F\|_{m, \varepsilon} t. \quad (32)$$

For all $s > 0$, for $m \geq 5$, one has

$$\|H_{t+s} F - H_s F\|_{m-4, \varepsilon} \leq C_m \|F\|_{m, \varepsilon} t. \quad (33)$$

2. *Let $m \geq 4$. There exists $C_m \in \mathbb{R}^+$ such that, for all $F \in S_m(\mathcal{B}, \varepsilon)$,*

$$\left\| \frac{H_t F - F}{t} - \frac{1}{2} \Delta F \right\|_{m-4, \varepsilon} \leq C_m \|F\|_{m, \varepsilon} t^{1/2}. \quad (34)$$

For all $s > 0$, for $m \geq 5$, one has

$$\left\| \frac{H_{t+s} F - H_s F}{t} - \frac{1}{2} \Delta H_s F \right\|_{m-5, \varepsilon} \leq C_m \|F\|_{m, \varepsilon} t^{1/2}. \quad (35)$$

Proof. Formula (27), integrated with respect to Y on B^2 , gives, for $k \leq m-1$:

$$\begin{aligned} \int_{B^2} \tilde{F}(X + Y) d\mu_{B^2, t}(Y) &= F(X) + \sum_{i=1}^{k-1} \frac{1}{i!} \sum_{J \in \Gamma^i, \delta \in \{0,1\}^i} \int_{B^2} \left(\prod_{r=1}^i \ell_{w_{j_r}^{\delta_r}}(Y) \right) d\mu_{B^2, t}(Y) \frac{\partial^i F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_i}^{\delta_i}}(X) \\ &+ \sum_{J \in \Gamma^k, \delta \in \{0,1\}^k} \int_{B^2} \left(\prod_{r=1}^k \ell_{w_{j_r}^{\delta_r}}(Y) \right) \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} \mathcal{P} \left(\frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}} \right) (X + sY) ds d\mu_{B^2, t}(Y). \end{aligned} \quad (36)$$

We denote by R_k the last term in the preceding formula. We have seen in subsection 4.2 that these functions admit L^1 norms, which allows us to exchange sums and integrals on B^2 . Using Wick's formula, we see that odd order terms are equal to 0. One can give a bound for the rest:

Lemma 5.8. For all $X \in H^2$, with $F \in S_m(\mathcal{B}, \varepsilon)$ and $k \leq m-1$, one has

$$|R_k(X)| \leq \frac{1}{\sqrt{\pi}} \frac{1}{k!} \|F\|_{m,\varepsilon} 2^{\frac{3k}{2}} t^{\frac{k}{2}} \Gamma\left(\frac{k+1}{2}\right) \left(\sum_{\Gamma} \varepsilon_j\right)^k.$$

Proof. Notice that $\mathcal{P}\left(\frac{\partial^k F}{\partial w_{j_1}^{\delta_1} \dots \partial w_{j_k}^{\delta_k}}\right)$ is bounded by $\|F\|_{m,\varepsilon} \varepsilon_{j_1} \dots \varepsilon_{j_k}$. One applies Hölder's formula to the product of ℓ functions and one sums over j_1, \dots, j_k . \square

Even order terms allow us to find (thanks to Wicks formula) the successive powers of the Laplace operator and we get

$$(H_t F)(X) = \int_{B^2} \tilde{F}(X+Y) d\mu_{B^2,t}(Y) = F(X) + \sum_{0 < 2p \leq k-1} \frac{1}{p!} \left(\frac{t}{2}\right)^p \Delta^p F(X) + R_k. \quad (37)$$

Let us prove the point about continuity. For $k=2$ and $m=3$, we can state the following result, since the rest is of order t :

$$\forall X \in H^2, \quad |H_t F(X) - F(X)| \leq C_2 \|F\|_{3,\varepsilon} t,$$

with $C_2 = 2(\sum \varepsilon_j)^2$.

This yields (32) when $m=3$. To treat the general case one uses induction, working with $\partial_u^\alpha \partial_v^\beta F$, where α and β have depth 1 at most and using Proposition 5.4. To obtain (33) one applies H_s to (32) (and loses one order of differentiability) and applies the semigroup property (Proposition 5.3).

Let us prove the point about differentiability. For $k=3$ and $m=4$, one can, in particular, obtain the following result since the rest is of order $t^{3/2}$:

$$\forall X \in H^2, \quad \left| \frac{H_t F(X) - F(X)}{t} - \frac{1}{2} \Delta F(X) \right| \leq C_3 \|F\|_{4,\varepsilon} t^{1/2},$$

with $C_3 = \frac{1}{\sqrt{\pi} 3!} 2^{9/2} \Gamma(2) (\sum \varepsilon_j)^3$. This gives (34) when $m=4$. To treat the general case one uses induction, working with $\partial_u^\alpha \partial_v^\beta F$, where α and β have depth 1 at most and using Proposition 5.4. To obtain (35) one applies H_s to (34), (and loses one order of differentiability) and applies the semigroup property (Proposition 5.3).

This completes the proof of Proposition 5.7 \square

We now can state the main result about the heat operator in S_m classes. For the sake of clarity, the two first points repeat former results of this subsection.

Theorem 5.9. Let ε be summable.

1. For $m \geq 1$, the operator H_t is continuous from $S_m(\mathcal{B}, \varepsilon)$ to $S_{m-1}(\mathcal{B}, \varepsilon)$ and for $m \geq 2$, the operator Δ is continuous from $S_m(\mathcal{B}, \varepsilon)$ to $S_{m-2}(\mathcal{B}, \varepsilon)$.
2. For $m \geq 3$, H_t and Δ commute: for all $F \in S_m(\mathcal{B}, \varepsilon)$, $\Delta H_t F = H_t \Delta F \in S_{m-3}(\mathcal{B}, \varepsilon)$.
3. Let $m \geq 6$ and $F \in S_m(\mathcal{B}, \varepsilon)$. The application $t \mapsto H_t F$ is C^1 from $[0, +\infty[$ in $S_{m-6}(\mathcal{B}, \varepsilon)$ and its derivative is $t \mapsto \frac{1}{2} H_t \Delta F$.

Proof. It remains to prove the last point. Set $\varphi(t) = H_t F \in S_{m-1}(\mathcal{B}, \varepsilon)$. According to the preceding proposition, φ is differentiable on $[0, +\infty[$ and $\varphi'(t) = \frac{1}{2} \Delta H_t F = \frac{1}{2} H_t \Delta F$. But $H_t \Delta F \in S_{m-3}(\mathcal{B}, \varepsilon) \subset S_{m-6}(\mathcal{B}, \varepsilon)$. Since $\Delta F \in S_{m-2}(\mathcal{B}, \varepsilon)$, an application of point 3 (about continuity) proves that $t \mapsto H_t \Delta F$ is continuous from $[0, +\infty[$ in $S_{m-6}(\mathcal{B}, \varepsilon)$. \square

Remark It is not necessary to write $\Delta_{\mathcal{B}}$, because of Remark 4.8.

5.3 The heat operator in the classes $S(Q_A)$

In this subsection, the operator A is self adjoint, nonnegative and trace class.

Lemma 5.10. Let $f \in S(Q_A)$. For all $m \in \mathbb{N}^*$ and all U_1, \dots, U_m , the application $g_{m,U} : x \mapsto d^m f(x)(U_1, \dots, U_m)$ belongs to $S(Q_A)$ and $\|g_{m,U}\|_{Q_A} \leq \|f\|_{Q_A} \prod_{j=1}^m Q(U_j)^{1/2}$.

The application $H_t f$ is differentiable on H and

$$d(H_t f)(x) \cdot y = \int_B \mathcal{P}(u \mapsto df(u) \cdot y)(x+z) d\mu_{B,t}(z) = (H_t g_{1,y})(x).$$

Moreover

$$|H_t f(x+y) - H_t f(x) - (H_t g_{1,y})(x)| \leq \frac{1}{2} \|f\|_{Q_A} Q_A(y).$$

Proof. One checks that $g_{m,U}$ is C^∞ and that, for all integer $k \geq 1$ and all $h_1, \dots, h_k \in H$,

$$d^k g(x) \cdot (h_1, \dots, h_k) = d^{m+k} f(x) \cdot (h_1, \dots, h_k, U_1, \dots, U_m).$$

This proves that $g_{m,U} \in S(Q_A)$.

For $x, y \in H$, Taylor's formula gives

$$\tau_y f(x) = f(x) + df(x) \cdot y + \int_0^1 (1-s) d^2 f(x+sy) \cdot y^2 ds.$$

We denote by $R_2(x, y)$ the last term of this sum. Since $\tau_y f, f$ and $x \mapsto df(x) \cdot y$ have stochastic extensions in $L^p(B, \mu_{B,t})$, so does $x \mapsto R_2(x, y)$. One gets

$$H_t f(x+y) = H_t f(x) + \int_B (\mathcal{P}(df(\cdot) \cdot y)(x+z) d\mu_{B,t}(z) + \int_B \widetilde{R_2}(x+z) d\mu_{B,t}(z).$$

One checks that the first integral gives a linear application with respect to y . The hypotheses on f prove its continuity and the bound on the rest. \square

By induction on the order m on can deduce the following result:

Proposition 5.11. *Let $f \in S(Q_A)$. For all $t > 0$, the application $H_t f$ belongs to $S(Q_A)$ and $\|H_t f\|_{Q_A} \leq \|f\|_{Q_A}$. Moreover, for all integer m and all x, y_1, \dots, y_m , one has, with the preceding notations,*

$$d^m(H_t f)(x) \cdot (y_1, \dots, y_m) = H_t(g_{m,y_1, \dots, y_m})(x).$$

We denote by $\Delta f(x) = \text{Tr}(d^2 f(x))$ the trace of the operator M_x satisfying $\langle M_x U, V \rangle = d^2 f(x)(U, V)$ for all vectors U, V of H . Its existence is ensured by the inequalities (19) and one can see it, too, as a sum of partial derivatives (with respect to an arbitrary orthonormal basis of H). One can state the following proposition:

Proposition 5.12. *If $f \in S(Q_A)$, then $\Delta f \in S(Q_A)$ with $\|\Delta f\|_{Q_A} \leq \text{Tr}(A)\|f\|_{Q_A}$. Moreover, for all $t > 0$,*

$$\Delta(H_t f)(x) = H_t(\Delta f)(x).$$

Proof. Let (e_j) be an orthonormal basis of H . One can write

$$\text{Tr}(d^2 f(x)) = \lim_{n \rightarrow \infty} \sum_{s=1}^n d^2 f(x) \cdot (e_s, e_s) = \lim_{n \rightarrow \infty} \sum_{s=1}^n g_{2,e_s,e_s}(x),$$

with the notations of Lemma 5.10. Then the series $\sum g_{2,e_s,e_s}$ converges in $S(Q_A)$ because $\|g_{2,e_s,e_s}\|_{Q_A} \leq \|f\|_{Q_A} \langle A e_s, e_s \rangle$ and A is trace class. Hence $\Delta f \in S(Q_A)$ with $\|\Delta f\|_{Q_A} \leq \text{Tr}(A)\|f\|_{Q_A}$. Since H_t is continuous on $S(Q_A)$, one has

$$\text{Tr}(d^2 H_t f(x)) = \lim_{n \rightarrow \infty} \sum_{s=1}^n d^2 H_t f(x) \cdot (e_s, e_s) = \lim_{n \rightarrow \infty} H_t \left(\sum_{s=1}^n g_{2,e_s,e_s} \right)(x) = H_t \left(\sum_{s=1}^{\infty} g_{2,e_s,e_s} \right)(x) = H_t(\text{Tr}(d^2 f))(x).$$

\square

Proposition 5.13. *For all $f \in S(Q_A)$, one has*

$$\lim_{t \rightarrow 0} \left\| \frac{H_t(f) - f}{t} - \frac{1}{2} \Delta f \right\|_{Q_A} = 0.$$

Moreover, for all $s > 0$, one has

$$\lim_{t \rightarrow 0} \frac{(H_{t+s}f) - H_s f}{t} = \frac{1}{2} \text{Tr}(d^2 H_s f) = \frac{1}{2} \Delta H_s f = \frac{1}{2} H_s \Delta f,$$

the convergence taking place in $S(Q_A)$.

□

Proof. Let $x \in H$. First prove that

$$\lim_{t \rightarrow 0} \frac{(H_t(f))(x) - f(x)}{t} = \frac{1}{2} \text{Tr}(d^2 f(x)) = \frac{1}{2} \Delta f(x). \quad (38)$$

For $y \in H$, Taylor's formula gives

$$f(x+y) = f(x) + \sum_{j=1}^k \frac{1}{j!} d^j f(x) \cdot y^j + \int_0^1 \frac{(1-s)^k}{k!} d^{k+1} f(x+sy) \cdot y^{k+1} ds.$$

We denote by $R_k(y)$ the last term of the sum just above. According to Remark 3.5, $\tau_x f$ has a stochastic extension $\tau_x \tilde{f}$ in $L^p(B, \mu_{B,h})$, with respect to the variable y . Indeed, f admits a stochastic extension for all p and Definition 3.6 implies that it is Lipschitz continuous. By subtraction, the rest R_k also admits a stochastic extension \widetilde{R}_k . This extension is bounded as follows:

Lemma 5.14. *Let $t > 0$, $p \in [1, +\infty[$. For $k \in \mathbb{N}^*$, one has*

$$\|\widetilde{R}_k\|_{L^p(B, \mu_{B,t})} \leq \frac{1}{(k+1)!} \|f\|_{Q_A} C(p(k+1))^{k+1} S^{\frac{k+1}{\alpha(p(k+1))}} t^{\frac{k+1}{2}},$$

with $S = \sum_j \lambda_j$.

Proof. Let $(E_n)_n$ be an increasing sequence of $\mathcal{F}(H)$, whose union is dense in H . Then

$$\begin{aligned} \|\widetilde{R}_k\|_{L^p(B, \mu_{B,t})} &\leq \|\widetilde{R}_k - R_k \circ \tilde{\pi}_{E_n}\|_{L^p(B, \mu_{B,t})} + \|R_k \circ \tilde{\pi}_{E_n}\|_{L^p(B, \mu_{B,t})} \\ &\leq \|\widetilde{R}_k - R_k \circ \tilde{\pi}_{E_n}\|_{L^p(B, \mu_{B,t})} + \|f\|_{Q_A} \|Q_A^{\frac{k+1}{2}} \circ \tilde{\pi}_{E_n}\|_{L^p(B, \mu_{B,t})} \end{aligned}$$

by definition of R_k . Remark 3.9 enables us to give an upper bound independent of n for the second term and to let n converge to infinity. □

One can then write, extending in $L^1(B, \mu_{B,t})$, according to Proposition 3.12 :

$$\int_B \tilde{f}(x+y) d\mu_{B,t}(y) = f(x) + \sum_{j=1}^k \int_B \mathcal{P} \left(y \mapsto \frac{1}{j!} d^j f(x) \cdot y^j \right) d\mu_{B,t}(y) + \int_B \widetilde{R}_k(y) d\mu_{B,t}(y),$$

where \mathcal{P} represents the passage to the stochastic extension. For $j \leq k$ one uses the L^1 convergence and formula (3) to obtain

$$\begin{aligned} \int_B \mathcal{P} \left(y \mapsto d^j f(x) \cdot y^j \right) d\mu_{B,t}(y) &= \lim_{n \rightarrow \infty} \int_B d^j f(x) \cdot \tilde{\pi}_{E_n}(y)^j d\mu_{B,t}(y) \\ &= \lim_{n \rightarrow \infty} \int_{E_n} d^j f(x) \cdot z^j d\mu_{E_n,t}(z), \end{aligned}$$

where $(E_n)_n$ is an increasing sequence of $\mathcal{F}(H)$, whose union is dense in H . For odd j , the terms are equal to 0. For even j , one takes an arbitrary orthonormal basis of E_n , $(e_s)_{1 \leq s \leq \dim(E_n)}$, and one checks that

$$\int_{E_n} d^2 f(x) \cdot z^2 d\mu_{E_n,t}(z) = \sum_{s=1}^{\dim(E_n)} t \frac{\partial^2 f}{\partial e_s^2}(x).$$

One then gets that, for any orthonormal basis of H ,

$$\int_B \mathcal{P} \left(y \mapsto d^2 f(x) \cdot y^2 \right) d\mu_{B,t}(y) = t \sum_{j \in \mathbb{N}} \frac{\partial^2 f}{\partial e_j^2}(x) = t \text{Tr}(d^2 f(x)).$$

Applying the former reasoning to $k = 3$ and using the upper bound of \widetilde{R}_3 in L^1 yield

$$\left| \frac{(H_t(f))(x) - f(x)}{t} - \frac{1}{2} \text{Tr}(d^2 f(x)) \right| \leq \|f\|_{Q_A} \frac{1}{4!} C(4)^4 S^{\frac{4}{\alpha(4)}} t,$$

which holds for all $x \in H$. This proves Formula (5.7). Replacing f by g_{m,y_1,\dots,y_m} in this inequality, we obtain, thanks to Lemma 5.10 and Proposition 5.11,

$$\begin{aligned} & \left| \frac{(d^m H_t(f)(x) \cdot (y_1, \dots, y_m) - d^m f(x) \cdot (y_1, \dots, y_m))}{t} - \frac{1}{2} d^m \text{Tr}(d^2 f(x)) \cdot (y_1, \dots, y_m) \right| \\ & \leq \|g_{m,y_1,\dots,y_m}\|_{Q_A} \frac{1}{4!} C(4)^4 S^{\frac{4}{\alpha(4)}} t \\ & \leq \|f\|_{Q_A} \prod_{i=1}^m Q_A(y_i)^{1/2} \frac{1}{4!} C(4)^4 S^{\frac{4}{\alpha(4)}} t. \end{aligned}$$

One then has

$$\left\| \frac{H_t f - f}{t} - \frac{1}{2} \Delta f \right\|_{Q_A} \leq \frac{1}{4!} C(4)^4 S^{\frac{4}{\alpha(4)}} t \|f\|_{Q_A},$$

which gives the convergence in $S(Q_A)$.

According to Proposition 5.11, H_s is continuous on $S(Q_A)$ and its norm is smaller than 1. The semigroup property (Proposition 5.3) gives

$$\left\| \frac{H_{t+s} f - H_s f}{t} - \frac{1}{2} H_s \Delta f \right\|_{Q_A} \leq \frac{1}{4!} C(4)^4 S^{\frac{4}{\alpha(4)}} t \|f\|_{Q_A},$$

which achieves the demonstration of Proposition 5.13, since H_s and Δ commute. \square

Lemma 5.15. *Let $f \in S(Q_A)$, $x \in H$ and let (e_n) be an arbitrary orthonormal basis of H . One denotes by $\frac{\partial}{\partial x_j}$ the differentiation in the direction of e_j . For all integer j one sets*

$$(\Delta^j) f(x) = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \right)^j f(x).$$

One has for all $h > 0$,

$$H_t f(x) = f(x) + \sum_{j=1}^N \frac{1}{j!} \left(\frac{t}{2} \right)^j \Delta^j f(x) + \int_B \tilde{R}_{2N+1}(y) d\mu_{B,t}(y),$$

with the upper bound of Lemma 5.14.

Proof. One reasons as in the preceding demonstration but one considers $k = 2N + 1$ instead of stopping at $k = 3$. For even j one has

$$\int_{E_n} d^j f(x) \cdot z^j d\mu_{E_n,t}(z) = \int_{E_n} j! \sum_{\alpha \in \mathbb{N}^{\dim(E_n)}, |\alpha|=j} \frac{1}{\alpha!} \frac{\partial^j f}{\partial z^\alpha} z^\alpha d\mu_{E_n,t}(z)$$

and the terms where a coordinate of the multiindex α is odd are equal to 0. The computation of the other terms gives the result, thanks to the equality

$$\int_{\mathbb{R}} y^{2p} d\mu_{\mathbb{R},1}(y) = \pi^{-1/2} 2^p \Gamma(p + 1/2).$$

\square

As a corollary of Propositions 5.12 and 5.13, one can state the following commutation result, which will be used later on to prove a covariance result.

Proposition 5.16. *Let φ be linear, continuous on H and such that $\varphi^* \varphi = \varphi \varphi^* = \text{Id}_H$. Let A be a linear application satisfying the hypotheses of Definition 3.6. For all $f \in S(Q_A)$, one can write*

$$\forall t \geq 0, (H_t f) \circ \varphi = H_t(f \circ \varphi). \quad (39)$$

Proof. One verifies that $f \circ \varphi$ (denoted by f_φ) is in $S(Q_{\varphi^* A \varphi})$, with

$$d^2 f_\varphi(x) \cdot (U, V) = d^2 f(\varphi(x)) \cdot (\varphi(U), \varphi(V)) = \langle \varphi^* M_{\varphi(x)}(f) \varphi U, V \rangle$$

and

$$\|f_\varphi\|_{Q_{\varphi^* A \varphi}} = \|f\|_{Q_A}.$$

(We still denote here by $\Delta f(x) = \text{Tr}(d^2 f(x))$ the trace of the operator M_x satisfying $\langle M_x U, V \rangle = d^2 f(x)(U, V)$ for all vectors U, V in H .) Moreover, the operator $\varphi^* M_{\varphi(x)}(f) \varphi$ is trace class and has the same trace as $M_{\varphi(x)}(f)$. Thus

$$\Delta(f_\varphi(x)) = \text{Tr}(d^2 f_\varphi(x)) = \text{Tr}(\varphi^* M_{\varphi(x)}(f) \varphi) = \text{Tr}(M_{\varphi(x)}(f)) = \text{Tr}(d^2 f(\varphi(x))) = (\Delta f)(\varphi(x)).$$

Applying 5.13 and the above remark to $f \circ \varphi$, one gets that

$$\lim_{t \rightarrow 0} \frac{H_t(f_\varphi) - f_\varphi}{t} = \frac{1}{2} \Delta(f_\varphi) = \frac{1}{2} (\Delta f) \circ \varphi \quad \text{in } S(Q_{\varphi^* A \varphi}).$$

Composing with φ^* , one obtains that

$$\lim_{t \rightarrow 0} \left(\frac{H_t(f_\varphi) - f_\varphi}{t} \right) \circ \varphi^* = \frac{1}{2} \Delta f = \lim_{t \rightarrow 0} \left(\frac{H_t(f - f \circ \varphi)}{t} \right) \quad \text{in } S(Q_A).$$

If one denotes by T_t the operator defined on $S(Q_A)$ by $T_t f = H_t(f \circ \varphi) \circ \varphi^*$, one can verify that (T_t) is a semigroup on $S(Q_A)$. Since both semigroups (T_t) and (H_t) have the same infinitesimal generator $\frac{1}{2} \Delta$, which is continuous on $S(Q_A)$ (Proposition 5.12), they are uniformly continuous and equal ([P], Theorems 1.2 and 1.3, Chapter 1). This achieves the proof. \square

We now can state the main result of this part. For the sake of clarity, the first points repeat former results of the same part.

Theorem 5.17. *Let A be a linear application on H satisfying the hypotheses of Definition 3.6. For all $f \in S(Q_A)$, $\Delta f \in S(Q_A)$ with $\|\Delta f\|_{Q_A} \leq \text{Tr}(A) \|f\|_{Q_A}$. Moreover, for all $t > 0$, $\Delta(H_t f)(x) = H_t(\Delta f)(x)$. The function $t \mapsto H_t f$ is C^∞ on $[0, \infty[$ with values in $S(Q_A)$, with*

$$\frac{d^m}{dt^m} H_t f = \left(\frac{1}{2} \Delta \right)^m H_t f.$$

For all $N \in \mathbb{N}^*$, one has

$$H_t f = f + \sum_{k=1}^N \frac{t^k}{k!} \left(\frac{1}{2} \Delta \right)^k f + t^{N+1} R_N(t),$$

where $R_N \in S(Q_A)$ is bounded independently of $t \in [0, 1]$.

Proof. The first point comes from Proposition 5.11. For the differentiability, according to Proposition 5.13, the result holds for $m = 1$. But then, since Δ commutes with H_t (Proposition 5.12), one concludes by induction on m .

For the second point, one applies one of Taylor's formulae to $t \mapsto H_t f$, which gives

$$\|R_N(t)\|_{Q_A} \leq \frac{1}{(N+1)!} \sup_{s \in [0, t]} \left\| H_s \left(\left(\frac{1}{2} \Delta \right)^{N+1} f \right) \right\|_{Q_A} \leq \frac{1}{(N+1)!} \left\| \left(\frac{1}{2} \Delta \right)^{N+1} f \right\|_{Q_A}$$

according to Proposition 5.11. \square

6 Appendix

We list here very general results used in the main part of the article.

Lemma 6.1. *Let (Ω, \mathcal{T}, m) be a measure space. Let $N \geq 2$ be an integer. For $i \leq N$, let f_i, g_i be functions on Ω with values in \mathbb{R} such that, for all $p \in [1, +\infty[$, $f_i \in L^p(\Omega, \mathcal{T}, m)$, $g_i \in L^p(\Omega, \mathcal{T}, m)$. For all $p \in [1, +\infty[$, set $M_p = \max_{1 \leq i \leq N} (\|g_i\|_p, \|f_i\|_p)$. Then for all $p \in [1, +\infty[$,*

$$\left\| \prod_{i=1}^N f_i - \prod_{i=1}^N g_i \right\|_p \leq (M_p)^{N-1} \sum_{k=1}^N \|f_k - g_k\|_{pN}. \quad (40)$$

More precisely, one has

$$\left\| \prod_{i=1}^N f_i - \prod_{i=1}^N g_i \right\|_p \leq \sum_{k=1}^N \left(\prod_{i=1}^{k-1} \|f_i\|_{pN} \prod_{i=k+1}^N \|g_i\|_{pN} \right) \|f_k - g_k\|_{pN}. \quad (41)$$

Proof. According to the general Hölder's inequality, the products belong to every L^p since $\|\prod_{i=1}^N g_i\|_p \leq \prod_{i=1}^N \|g_i\|_{pN}$ (for example). One decomposes $\prod_{i=1}^N f_i - \prod_{i=1}^N g_i$ as

$$\prod_{i=1}^N f_i - \prod_{i=1}^N g_i = \sum_{k=1}^N \left(\prod_{i=1}^k f_i \prod_{i=k+1}^N g_i - \prod_{i=1}^{k-1} f_i \prod_{i=k}^N g_i \right) = \sum_{k=1}^N \left(\prod_{i=1}^{k-1} f_i \prod_{i=k+1}^N g_i \right) (f_k - g_k)$$

agreeing that a product on an empty set of indices is equal to 1. Hölder's inequality gives

$$\left\| \left(\prod_{i=1}^{k-1} f_i \prod_{i=k+1}^N g_i \right) (f_k - g_k) \right\|_p \leq \prod_{i=1}^{k-1} \|f_i\|_{pN} \prod_{i=k+1}^N \|g_i\|_{pN} \|f_k - g_k\|_{pN}.$$

The first $N-1$ factors are smaller than M_{pN} . Taking the sum gives the result. \square

Corollary 6.2. *Let F_1, \dots, F_N be functions defined on H^2 and admitting stochastic extensions $\widetilde{F}_1, \dots, \widetilde{F}_N$ in $L^p(B^2, \mu_{B^2, h})$ for all $p \in [1, +\infty[$. Then $\prod_{i=1}^N F_i$ admits $\prod_{i=1}^N \widetilde{F}_i$ as a stochastic extension in $L^p(B^2, \mu_{B^2, h})$ for all $p \in [1, +\infty[$.*

Proof. Let (E_n) be an increasing sequence of $\mathcal{F}(H^2)$, whose union is dense in H^2 . According to (40),

$$\left\| \prod_{i=1}^N F_i \circ \widetilde{\pi}_{E_n} - \prod_{i=1}^N \widetilde{F}_i \right\|_p \leq \left(\sup_{n \in \mathbb{N}, i \leq N} (\|F_i \circ \widetilde{\pi}_{E_n}\|_{Np}, \|\widetilde{F}_i\|_{Np}) \right)^{N-1} \sum_{i=1}^N \|F_i \circ \widetilde{\pi}_{E_n} - \widetilde{F}_i\|_{Np},$$

which gives the result. \square

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